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OCTOBER 1959

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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THE ROLE OF NUMERICAL ANALYSIS IN AN UNDERGRADUATE PROGRAM*

GEORGE E. FORSYTHE, Stanford University

1. Where do mathematics students go? Before considering the aim of undergraduate mathematics education we should recall approximately who the students are who take college mathematics courses, and where they are going. I assume that everywhere, as at Stanford, most mathematics students major in other fields. Usually students of science and engineering form most of our clientele. The increasing role of mathematics in economics, psychology, etc., is bringing a number of serious students of social science. And we have an increasing number of mathematics majors—students who are naturally dearest to our hearts.

It should be obvious that the engineers and other nonmajors are primarily interested in the *application* of mathematics, and not in its *structure*. The best of the nonmajors will apply mathematical analysis in their professions. A large number of the less gifted, particularly among the engineers, will face a considerable number of computations in their future work, and this may represent the main future use of their mathematical background.

And our majors, where are they bound? One wishes in vain that all the best of them would either teach high school or take graduate work. Of those who attain the Ph.D., about half go into industry and apply mathematics in various areas. Almost all of them will be connected to some degree with automatic computation. But the majority of our undergraduate mathematics majors are lured at once into the market place, where they are greatly in demand as servants of the fast-multiplying family of fast-multiplying computers.

Why are they in such demand? There seem to be over 3000 automatic digital computers now installed in the United States, with more on the way. As a rough estimate, each automatic computer needs to have 10 attendants who serve it as mathematicians—programmers, coders, analysts, supervisors, etc. The resulting requirement for 30,000 computer mathematicians should be compared with the combined membership of the American Mathematical Society, Mathematical Association of America, Society for Industrial and Applied Mathematics, Association for Computing Machinery, Institute of Mathematical Statistics, and American Statistical Association—under 20,000 persons. While some makeshift arrangements are possible, the disparity in numbers is creating the unprecedented demand (and salary) for the new A.B. in mathematics.

It must be noted as a digression that as a practitioner of mathematics, the new A.B. is not in a strong position, despite industry's demand for him. He knows little of what he needs to do a good job—superficially because his teachers have not prepared him to be a practitioner of mathematics, but fundamentally

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because of the brevity of his mathematical education. As a mathematically half-educated man he needs to learn both caution and confidence. But, if he is confident, wise in human relations, aggressive, and lucky, he will become a supervisor of coders—a position in which with his half education he sometimes does serious damage to his company and the reputation of scientific computing. As far as technical advancement is concerned, coding appears to be a terminal occupation, except to those with considerable graduate work in mathematics.

For discussions of mathematicians in the market place, see Rees [12] and Fry [5].

In summary, a very large number of undergraduate students of mathematics are going to apply mathematics in their future work, and much of this application will involve automatic computation. How should this influence our program?

2. Aims of undergraduate mathematics education. In the author's opinion, we should have the following goals for the mathematical education of our undergraduates, whether majors or not.

a. The student should above all learn as much as possible of the structure, significant ideas, and points of view which constitute mathematics. This is the *sine qua non.*

b. The student should learn to read independently the mathematical literature at his level of comprehension, and to use mathematical language to write facts, proofs, and ideas precisely and unambiguously, in a style acceptable to not-too-strict editors of mathematical journals. With the growth of applied mathematics, a good many papers are being written by the A.B. in mathematics; it is unfair to the authors, to editors, and to readers that so many of their authors write as mathematical illiterates.

c. The student should know the tools of the mathematical profession (books and machines), and how and where to find them and to keep up to date on them. Within his general limitations, he should be competent in the library. He should know roughly what bibliographies exist, what tables of useful functions there are, what machines can and cannot do, and so on. Since he is fully competent to learn it, he should be able to code one automatic computing machine quite well. Finally, he should have a general idea of what practitioners of pure and applied mathematics there are, and their abilities. For example, we must teach the future engineer to consult with statisticians, numerical analysts, and other experts at an appropriately early time, instead of trying to go it alone until an emergency arises.

d. The student should have cultivated and practiced the solution of mathematical problems new to him. This is where research ability is built. It hardly matters whether the problems have been solved before by some one else.

e. The student should have gone fairly deeply into some other field of knowledge—preferably some field where mathematics is used, like a physical science or the mathematical aspects of economics. He should have acquired as

much experience as possible in communicating with practitioners of this other field, so as to learn to translate mathematical terms into their language, and vice versa. Here he should have struggled with the mathematical idealization of problems from the real world.

f. The student should have learned to enjoy mathematical study.

It seems to me that these constitute valid goals also for mathematics in grade school, secondary school, and graduate school, if the standards of writing and item *c* are modified appropriately. In graduate school one might wish to add:

g. The student should receive some supervised experience in serving as a mathematical consultant.

3. The abstract versus the concrete. In this post-sputnik era one hears and reads a great deal of criticism of mathematical education at all levels, but particularly in high schools. Many, perhaps most, mathematical critics deplore the lack of rigor and abstraction, and the lack of understanding of the mathematical structure of algebra, geometry, and other subjects. I believe that criticism to be valid. But I have a different complaint—that much teaching is not intuitive enough nor well enough illustrated. I wish mathematics classes would get down to earth more often and make actual application of newly learned theorems to concrete problems of a realistic nature.

In these ideas I agree with Felix Klein [9], and quote from his book, pages 15 and 16: "I have already emphasized the fact that, in the schools, applications accompany arithmetic from the beginning, that the pupil learns not only to understand the rules, but to do something with them. And it should always be so in the teaching of mathematics! Of course, the logical connections, one might say *the rigid skeleton in the mathematical organism*, must remain, in order to give it its peculiar trustworthiness. But the living thing in mathematics, its most important stimulus, its effectiveness in all directions, depends entirely ["entirely" is too strong—G.E.F.] upon the applications, *i.e.*, upon the mutual relations between those purely logical things and all other domains. To banish applications from mathematics would be comparable to seeking the essence of the living animal in the skeleton alone, without considering muscles, nerves and tissues, instincts, in short, the very life of the animal." Klein goes on to say that, while in research differentiation between pure and applied mathematicians may be essential, such differentiation is not reasonable in teaching.

It is not contradictory to criticize in the same breath mathematics teaching for not being abstract enough and for not being concrete enough. I think both these criticisms are frequently valid. Taken together, they imply a *hazy* mathematics course which has neither the beautiful structure of a logical entity nor fascinating application to the physical world. To the extent that mathematics courses are hazy and shapeless, the author claims there are two directions in which reform is needed—to strengthen the skeleton, and to add a variety of applications.

For example, a student with a poor knowledge of high school algebra is weak

on structure—for example, he is unaware that the whole subject flows from a few postulates and principles, or that proofs are just as relevant to algebra as to geometry. But he is also weak in the application of algebra. He is notably unable to convert word problems into algebraic form—always a hard part of algebra. He is also weak in arithmetic, and lacks good intuition about what happens when one gives variables numerical values. (Can $-x$ really be positive?)

The use of high school algebra is purely for an example. Quite parallel instances occur everywhere in school and college.

It must be emphasized that the concrete application of an abstract theory is rarely in practice just a special case of the abstract theory! To illustrate this paradox, consider the multiplication of real numbers x, y. We know that $xy = yx$, and that $x(yz) = (xy)z$. However, in the concrete application to digital computers, one is forced to curtail the decimal (or binary) expansions of numbers at some fixed precision. Thus, x, y, z are represented by *digital* numbers $\bar{x}, \bar{y}, \bar{z}$. Moreover, exact multiplication of digital numbers does not lead to a digital number, and so one must be content with a *pseudomultiplication* $\times$, where the pseudoproduct is digital. Difficulties arise from the fact that often $\bar{x} \times \bar{y} \neq \bar{y} \times \bar{x}$ and $\bar{x} \times (\bar{y} \times \bar{z}) \neq (\bar{x} \times \bar{y}) \times \bar{z}$. This is perhaps a typical example of essentially new problems created by concrete applications of a mathematical theory. The problems may in turn lead to a new mathematical theory to solve them.

Since my personal interest is in the concrete side of mathematics, I shall concentrate on it in the following, leaving structure to others more interested and better qualified. For other opinions on the teaching of concrete mathematics, see Tukey [14] and Weyl [15]. Tukey refers to *monastic* and *secular* mathematics.

4. What is numerical analysis? Numerical analysis (apparently a postwar term) is a branch of applied mathematics, namely the art and science of using digital computation to solve scientific problems. Traditionally numerical analysis included such matters as numerical quadrature and the solution of polynomial equations and ordinary differential equations. Since automatic digital computing machines are newly on the scene, the part of numerical analysis dealing with their use in scientific computation is the live area for research. And, in practice, the large machines are dominating scientific computation in this country. Hence the problems of using such machines constitute most of what I consider to be numerical analysis today. Since the machines can deal with vast quantities of data, the solution of *large* problems is frequently asked. For example, one thinks today of solving linear algebraic systems with 100 to 1000 unknowns, where, with a desk calculator, one with 10 unknowns requires most of a day. But, even more than size, the automatic nature of these machines has changed the character of problems. We now consider that numerical analysis covers any type of problem which an automatic computer should be able to solve—any problem which can be reduced to an algorithm or to a flow chart. At this point we consider such things as formal integration of a rational function

as numerical analysis. In fact, most exercises other than "word problems" and proofs which occur in undergraduate courses involve some algorithm, and may be considered as part of numerical analysis!

Numerical analysis has different branches. Among these are the design of algorithms, the analysis of errors, and techniques for coding (mainly the use of machines to prepare codes for machines).

5. What numerical analysis should the undergraduate study? The amount of analysis and algebra at the command of the undergraduate limits the depth to which he can pursue numerical analysis, and in particular error analysis. This tends to force the undergraduate numerical analysis course into a collection of recipes (simple algorithms). But this is hardly an excuse for a course, because the student should be well enough educated to be able to read and apply algorithms for himself. To define an appropriate syllabus for undergraduate numerical analysis is therefore a bit delicate. I consider the following to be about right:

a. He should learn to code one automatic digital computing machine. Actual experience operating a machine is certainly valuable and stimulating, but it is not essential. One can learn to code any machine without seeing it, and indeed all machines are coded before they are built. Except for the specialist, it will suffice to learn to code in an algebraic language like FORTRAN, which can be transformed by special translator codes into machine-language codes. A standard international algebraic language for computers, called ALGOL, is now being developed.

b. He should get a feeling for what automatic computers can do, and what they cannot do. He should learn that they are no substitute for creative thought, and yet that they can do a good deal of what passes for thought in this world. The student must learn the distinction. He should also learn that collaboration with an automatic computer compels precise formulation of the problem, and 100 percent accuracy in preparation of the code. In this respect the automatic computer really forces that precision of thinking which is alleged to be a product of any study of mathematics.

c. He must learn the tools of the computer—what tables, machines, books, codes, bibliographies, and people exist to help him solve future computing problems.

d. He should learn a few typical algorithms, including, for example, Newton's method for finding zeros, Crout's method for solving linear systems, and Euler's method for ordinary differential equations. If not too many algorithms are undertaken, it should be possible to study each fairly deeply.

e. He should have been exposed to some instances of treachery in computing, enough to teach him great caution. For example, he might find out that, whereas the polynomial

$$F(x) = (x - 1)(x - 2) \cdots (x - 20) = x^{20} - 210x^{19} + \cdots + 20!$$

has zeros 1, 2, $\cdots$, 20, the slightly rounded polynomial $F(x)-2^{-23}x^{19}$ has among its zeros numbers near 20.846 and $13.99 \pm 2.5i$ (Wilkinson [16]). This example should keep any one from using polynomial-solving routines blindly! We hope it would teach the student that you still have to think in the era of automatic computers.

f. He should have some exposure to careful error analysis. For example, he might have studied Wilkinson's round-off error analysis of Crout's method [17]. A fruitful point of view here is to see how much one has to change the data to make the computed answer correct, rather than to ask how wrong the answer is for the given data.

g. He should have a firsthand acquaintance with arithmetical operations on numbers, gained from desk computation and also, if possible, from work with an automatic computer.

Professor Perlis has remarked that, whereas we *think* we know something when we learn it, and are *convinced* we know it when we can teach it, the fact is that we don't *really* know it until we can code it for an automatic computer!

6. How can we teach this numerical analysis? There seem to be two ways of teaching numerical analysis within the course structure of the American college. One is to teach special courses in it, and the other is to mix it into other relevant mathematics courses. Obviously this year's choice between these methods depends on the present existence of teaching materials and trained staff. But one may also ask which would be the preferable solution, assuming that teaching materials and staff will eventually support the prevailing decision.

Many professional numerical analysts, the author included, agree that numerical analysis should be mixed into most undergraduate mathematics courses. From our point of view, many undergraduate exercises in such subjects as calculus, linear algebra, and differential equations are in fact solvable with automatic computers, and hence belong to numerical analysis. We think their real importance is the illumination of mathematical theory, and hence that such exercises should stay where they are now: in the theoretical courses. However, to pick one example from linear algebra, we feel that the conventional system of 3 linear equations in 3 unknowns with small integral coefficients does not furnish a sufficiently bright illumination of linear algebraic theory. And it certainly fails dismally as a fair illustration of linear algebra in industry today. We therefore feel that more realistic examples should be considered in the linear algebra course. One could easily consider 10 or 20 equations in 10 or 20 unknowns, and expect students to produce a complete flow chart, or a code in some algebraic language like ALGOL.

Turning to ordinary differential equations, we feel that no one understands the real nature of a differential equation $dy/dx=f(x, y)$ until he has experimented with computational or graphical procedures for its solution. Such a simple process as Euler's method, while not accurate, is both useful and worthy of study. What normally passes for numerical methods in differential equations

courses bears little resemblance to how such equations are solved in practice. Why not mix realistic practice right with the theory, where they can influence and strengthen each other?

There is one exception, a subject sufficiently demanding and sufficiently remote from other undergraduate mathematics to warrant a special course. This is coding. As we have indicated, we think every undergraduate mathematics student should know how to code some machine fairly well. (I would also include all undergraduate students, for I feel that the computer revolution will have such a great impact on all our lives that every college graduate should understand it intimately. Possibly it will eventually be taught in the ninth grade for the same reason.) Since coding presupposes no mathematics beyond arithmetic, it can be taught to freshmen. I recommend a two-hour-per-week semester course in coding, to be taken as early as possible. There are texts already available for such a course, including one by Andree [1]. The coding course should also include the material in c of Section 5—an exposure to bibliographies, tables, codes, and other library material. For, if numerical analysis is integrated with other mathematics, the student may nowhere else be taught by specialists in numerical analysis.

Until a student actually has a digital computer to work with, the greatest benefit of a coding course is the *point of view*. The student will learn what constitutes creative thought, and what is merely routine. The *flow chart*, almost essential to coding, is one of the greater tools in scientific and technological thinking. It probably originated in the block diagram of the electrical engineer. It would be a useful device in mathematical teaching and thinking. For example, for learning the basic concept of functions of one variable and their composition, the following block diagram seems far superior to the less graphic functional notation $y=f(g(x))$:

$$x \rightarrow \boxed{g} \rightarrow g(x) \rightarrow \boxed{f} \rightarrow y.$$

With coding behind him, the student can analyze many algorithms from the point of view of preparing an actual or potential code to solve them. And, to the extent that he has a machine available, he can solve realistic problems in any area.

I should like to point out that numerical analysis was mixed into a theoretical course in mathematics by Dickson in his book [3]. Such a course in the theory of equations was taught for years in American colleges, and is only now beginning to disappear in favor of more linear algebra. Dickson and other authors following his lead included Horner's and Newton's root-solving procedures, evaluation of determinants, solution of linear algebraic systems, and so forth. With the evolution of computing equipment and numerical analysis, much of Dickson's explicit and implicit advice on numerical analysis became obsolete and misleading, but for many of us it was the only numerical analysis we ever saw in college.

Let us consider the common practice of teaching one or more separate courses

in numerical analysis for undergraduates. These courses have not required coding as a prerequisite, although this is changing now. Judging from the textbooks, some of these courses tend to be filled more with algorithms than with analysis, and thus may be shallow for competent students. Milne's book [11] is excellent, but most of it could be read by the student whenever he needed a certain technique. The basic mathematical concepts of the book, like orthogonal polynomial systems and the representation of linear functionals, should really be covered in other courses.

The other extreme in a course is represented by Hildebrand's beautiful book [7]. This includes real analysis of round-off errors, truncation errors, and so on, and is a great pleasure to a numerical analyst. But the typical good undergraduate is simply not enough of an analyst to do justice to the course, if I can judge from Stanford and U.C.L.A. While the instructor can teach the analysis as he goes along, it is questionable whether a special numerical analysis course is the correct place for this.

In at least one university (U.C.L.A.) there are four undergraduate numerical analysis courses covering the fields of interpolation, linear algebra, ordinary differential equations, and approximation, respectively. These are excellent courses, and there seems to be no real reason for not merging most of them with basic undergraduate courses in corresponding subjects—say advanced calculus, linear algebra, ordinary differential equations, and Fourier series. Both sets of courses would profit from the merger. If coding were common knowledge of all mathematics students, there would seem to be no necessary obstacle to the merger.

As nearly as one can tell from Gnedenko's article [6], the Russian practice seems to be to include numerical analysis problems in a mathematical laboratory. In his article Gnedenko lists several required problems in numerical analysis. These problems are large ones, and require plenty of both analysis and computation. How the laboratory is coordinated with the mathematical lectures, we have no idea.

In summary, I favor a special coding course for all students, to be taken early. I should like to see material prepared so that numerical analysis could otherwise be sifted in with the basic mathematical theory, as illustrations of the theory and as motivation for more theory. In the next section are examples of where this mixing might occur.

7. Examples of numerical analysis mixed into other courses. In this section I shall outline places where numerical analysis comes into standard courses in college mathematics, and in particular where problems should be treated with flow charts or codes. Naturally these vary from small problems to quite large ones.

Analytical geometry. Almost any exercise, like finding the bisector of the angle between two given lines $ax+by+c=0$ and $a_1x+b_1y+c_1=0$, where the constants are arbitrary.

Finding areas and volumes included by lines or planes.

Testing for concurrence or collinearity.

Determining the type of a conic corresponding to a second-degree equation in x and y.

Calculus. Formal integration, *e.g.*, making a flow chart for integrating a general rational function of x of degree no higher than 6, considering all cases.

Studying the asymptotic convergence of Newton's process for getting real zeros of a function. What about multiple roots? Bounding the error in a presumed root.

Use of differentials to get the linearized expression for the error in evaluating a polynomial. Use of polynomials of degrees like 20 or 100 for examples.

Study of sequences of difference quotients whose limit is dy/dx. Accelerating the convergence of the sequences. Central differences versus one-sided differences.

Numerical quadrature on an automatic computer. Study of sequences of quadratures by the trapezoidal rule, as $h \to 0$.

The error in the trapezoidal rule, when $f''(x)$ is integrable, or not integrable.

Upper and lower bounds for integrals. Study of Riemann sums as a sequence to be accelerated to its limit, the Riemann integral.

Effect of rounding numbers on the sharp discontinuity in the formula for $\int x^n dx$ when $n = -1$. See Forsythe [4].

Algebra. Number of multiplications required to evaluate a polynomial of degree n. See Todd [13].

Rapidly computable definition of determinant of A. (The definition in terms of $n!$ terms is impossibly slow for large n.)

Evaluation of determinants by row elimination. Importance of choosing large pivotal elements. Methods of doing so conveniently with matrices of order 1000 or more, stored on magnetic tape.

Coding the calculation of the row rank of a set of row vectors.

Computation of $\sqrt{a+ib}$ on a computer, with minimum error.

Actually solving linear algebraic systems. Round-off errors.

Actually computing eigenvalues of matrices. The great distinction between symmetric and nonsymmetric matrices in this problem.

Determination of whether a quadratic form is definite.

Solution of linear inequalities. Duality theorems, and their use in computation.

"Finite calculus" of sums and differences.

Convergence of linear iterative processes.

Bounding the round-off errors in pseudoarithmetic operations; see Section 3 above.

Logic. Preparing a code to test the truth or falsity of a class of propositions. All sorts of questions relating computability to the theory of recursive functions.

Advanced calculus. The nature of the remainder in Taylor's theorem.

Interpolation theory and remainders. Convergence of interpolation polynomials.

Direct methods in calculation of minimizing functions in calculus of variations.

Numerical summation of series with a computer. Code for computing $\sum_{n=0}^{\infty} f(n)x^n$, where $f(n)$ is a polynomial of low order, and $|x|<1$.

Ordinary differential equations. Coding and running Euler's method for solving $dy/dx=f(x, y)$. Bounding its error, and comparing with experience on test problems.

Better methods for numerical integration of same differential equation.

Two-point boundary-value problems.

Computing the solution of an initial-value problem for $y''+c_1y'+c_2y=0$, where the characteristic equation nearly has a double root, and where the coefficients are 10-decimal numbers; see [4].

Probability and Statistics. Monte Carlo method for integration. Stratified sampling methods for greater accuracy.

Generating normal random processes on a digital computer, with a prescribed power spectrum density.

Estimating the power spectral density from an observed instance of a random process.

Use of automatic computer to find mean and variance of given data which are very numerous.

Number theory. Generation and testing of pseudorandom numbers for use in Monte Carlo experiments.

Tests for primality of large numbers.

All kinds of combinatorial problems.

Geometry. Iteration of rational transformations—theory and practice.

Study of gradient methods on a computer.

Fourier series. Numerical summation of a given series; acceleration of convergence. Orthogonal polynomials over finite point sets, and their use in curve fitting.

8. Implementing the recommendations. Since it does not seem practical to teach an undergraduate course without following a textbook fairly closely, courses are likely to change only when textbooks change. There is no textbook for a modern integrated version of a pure mathematical field and its numerical application. Moreover, of all the numerical analysis texts published in the past 10 years, only one seems really concerned with the problems of automatic digital computation. While desk calculators are far from dead, existing textbooks make it possible for any one to learn desk-calculator technique very well indeed. Future textbooks in numerical analysis must emphasize the use of automatic machinery. The one book to deal with automatic computation is the recent

Modern Computing Methods [2], regrettably anonymous but actually written by L. Fox, E. T. Goodwin, F. W. J. Olver, and J. H. Wilkinson. This is very clear, too short to be comprehensive, but full of annotated references to major sources.

I therefore seem to be making a recommendation which is almost two generations ahead of the current textbooks. The first generation might follow *Modern Computing Methods* and treat various aspects of computation by automatic computers. The second generation might integrate this material into basic existing mathematics courses. One can hope that some books will leap the whole step at once.

There is one method of accelerating the appearance of textbooks in which mathematical theory and numerical applications are suitably intertwined—namely to write applied supplements to existing theoretical texts. For example, one might prepare a companion text to a textbook in the theory of linear algebra, in which numerical methods are discussed from the point of view of automatic computation, with suitable problems. Such a companion text might presume that the student knows how to code, but that he is learning the theory of linear algebra for the first time from the basic textbook. The companion text would refer to the basic text for theory, and would indicate the appropriate stages at which the companion book should be used.

Such companion texts would be useful for at least the following subjects: calculus, differential equations, statistics, linear algebra, advanced calculus. It would be well to include suitable analytical applications, as well as numerical ones.

My basic recommendation for integrated courses in mathematical theory and application may have to await the appearance of suitable integrated texts or companion texts. What can be done meanwhile, aside from urging one's friends to write expositions?

For one thing, the coding course can be initiated at once anywhere. One needs only a textbook, say Andree [1] or McCracken [10], plus coding manuals which the International Business Machines Corporation or other computer manufacturers would probably be glad to furnish through their local representatives. McCracken's text deals with a mythical machine.

No matter what courses are given, I believe all students of mathematics should have easy access to semiautomatic or fully automatic desk calculating machines, and some problems to solve with their aid. Even such an apparently simple problem as using an 8-bank calculator to find close upper and lower bounds for

$$1.56898x^3 - 0.45678x^2 + 9.99938x - 0.42057 \qquad \text{for } x = \pi/2$$

will give students a surprising amount of insight into some problems of numerical calculation. Some experience with desk calculators is almost an indispensable prerequisite to use of an automatic computer.

If there is an automatic computer, students should be encouraged to use it

to find upper and lower bounds for various quantities which they deal with in their several theoretical courses. Such exercises should develop a good deal of power in numerical analysis.

Finally, until integrated texts are available, it will probably be profitable to offer a specialized course in numerical analysis to undergraduates, using as a text such existing books as *Modern Computing Methods* [2], Hildebrand [7], Milne [11], or—better yet—selections from each. There are other good textbooks also. Such a course should include as exercises, problems which I envision as later appearing in several theoretical courses.

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A CONNECTED TOPOLOGY FOR THE INTEGERS

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A topology D for the positive integers is obtained when those arithmetic progressions $\{an+b\}$ with $(a, b)=1$ are taken as a basis for the open sets. They form a basis because the intersection of two such progressions is of the same type, or empty, as is easily verified. Note that every nonempty open set, being a union of arithmetic progressions, must be infinite.

This topology furnishes an interesting proof of

THEOREM 1. *The number of primes is infinite.*

Proof. If p is prime, the progression $\{np\}$ is closed, since its complement is $\{np+1\}\cup\{np+2\}\cup\cdots\cup\{np+(p-1)\}$, a union of open sets. Consider the union $X=\cup_p\{np\}$ extended over all primes. If this is a finite union of closed sets, then X is closed. But the complement of X is $\{1\}$, which is neither empty nor infinite. Since the complement of X is not open, X cannot be closed, the union is not a finite one, and the number of primes is infinite.

(A similar proof, in a stronger and very disconnected topology, was given by Furstenberg [2].)

THEOREM 2. *The topology D is Hausdorff.*

Proof. Given distinct positive integers s and t, choose a prime p (by Theorem 1) which exceeds max (s, t). Then $\{pn+s\}$ and $\{pn+t\}$ are disjoint open sets which separate s and t.

THEOREM 3. *The topology D is connected.*

Proof. Suppose the integers could be represented as the union of two disjoint nonempty open sets O_1 and O_2. Let $\{a_1n+b_1\}$ be a basis set in O_1, and let $\{a_2n+b_2\}$ be a basis set in O_2. Let α be a multiple of a_1. If α were in O_2, we would have $\alpha=An_0+B$, where $\{An+B\}\subset O_2$. Since $(A, B)=1$, we would have $(\alpha, A)=1$, and hence $(a_1, A)=1$. But then $\{a_1n+b_1\}$ and $\{An+B\}$ would intersect infinitely often, contradicting disjointness of O_1 and O_2. Thus all multiples of a_1 must belong to O_1. Similarly the multiples of a_2 must belong to O_2. But then the *common* multiples of a_1 and a_2 must belong to both O_1 and O_2, which contradicts disjointness.

The author has recently learned that a proof of the connectedness of the topology D, without reference to number theory, was presented by Morton Brown at the April 1953 meeting of the American Mathematical Society in New York [1].

THEOREM 4. *The topology D is not regular.*

Proof. Suppose that open coverings are given for the closed set $\{2n\}$ and for the point $\{1\}$ outside it. Any open covering of $\{1\}$ not intersecting $\{2n\}$

must include a progression $\{en+1\}$, where e is even. That is, $e \in \{2n\}$. Let $\{an+b\}$ be the member of the open covering of $\{2n\}$ which contains e, so that $e=an_0+b$. Since $(a, b)=1$, we have $(a, e)=1$, whereby $\{an+b\}$ intersects $\{en+1\}$ infinitely often. Thus the closed set $\{2n\}$ and the point $\{1\}$ cannot have disjoint open neighborhoods.

THEOREM 5. *The topology D is not compact.*

Proof. The union $\cup_p \{np-1\}$ extended over all primes is an infinite open covering for the positive integers. Since the omission of any progression $\{nq-1\}$ leaves the number $q-1$ uncovered, the Heine-Borel property fails.

Actually, the topology D is not even locally compact, because every locally compact Hausdorff space is regular. For a proof of this, as well as for the more basic definitions of point-set topology, the reader is referred to [5].

Dirichlet's theorem, which asserts that every progression $\{an+b\}$ with $(a, b)=1$ contains infinitely many primes, has an elegant formulation in terms of the topology D.

THEOREM 6. *Dirichlet's theorem is equivalent to the assertion that the primes are a dense subset of the integers in the topology D.*

Proof. Assume first the validity of Dirichlet's theorem. Then every nonempty open set contains primes, so that the primes are a dense subset of the integers. Conversely, assume that the primes are a dense subset. Then every nonempty open set, and in particular all the progressions $\{an+b\}$ with $(a, b)=1$, must contain primes. It is well known [4] that if every such progression contains at least one prime, then every such progression contains infinitely many primes. (In topological terminology: "If the closure of the primes is the integers, then the derived set of the primes is the integers.")

It appears unlikely that a complete topological proof of Dirichlet's theorem can be given along these lines without the introduction of powerful new ideas and methods.

Another familiar fact capable of topological formulation is

THEOREM 7. *In the topology D, the interior of the set of primes is empty.*

Proof. If there were an open set consisting entirely of primes, there would be a progression $\{an+b\}$ with $1 \leqq b \leqq a$ consisting entirely of primes. But with $n_0=a+b+1$, $an_0+b=(a+b)(a+1)$, which is composite.

It is interesting to consider also the topology D' for the positive integers, which has as a basis those progressions $\{an+b\}$ with $(a, b)=1$ for all $n>N$. (Here N is allowed to assume all values.) This topology is clearly *stronger* than D, although Theorems 1 to 7 are still valid in D'. Moreover, certain theorems related to Eratosthenes' sieve are available in D'. In particular,

THEOREM 8. *The set of positive integers m such that $6m-1$ and $6m+1$ are a pair of "prime twins" is closed in D'.*

Proof. It is known [3] that the numbers m in question are precisely those positive integers *not* expressible in the form $6ab \pm a \pm b$ for any $a \geqq 1$ and $b \geqq 1$. Thus the *complement* of our set is $\bigcup_{b \geq 1} \{(6b \pm 1)a \pm b\}$, where each progression is restricted to $a \geqq 1$, and is open because $(6b \pm 1, b) = 1$. The union is open in D', because it is a union of open sets. Thus the integers m for which $6m-1$ and $6m+1$ are both prime form a closed set.

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AN APPLICATION OF TURNS AND SLIDES TO SPHERICAL GEOMETRY

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1. Introduction. The ordinary or Euler spherical triangle, the sides and angles of which are assumed to lie between 0 and π, is determined uniquely by three points on a sphere, provided no two of them are diametrically opposite. Since each side of an Euler triangle has two poles, the choice of a polar triangle is ambiguous; moreover, more than one Euler triangle may exist with parts satisfying the law of cosines on a sphere. Moebius [3], consequently, generalized the concept of a spherical triangle in order to remove these ambiguities by orienting the sides and the angles and allowing them to assume all real values between 0 and 2π. Moebius's device, though adequate for a satisfactory interpretation of the formulas involving the sines and cosines of the sides and angles, did not go far enough. Such formulas Felix Klein [2] called formulas of the first kind to distinguish them from others, essentially different, connecting the sines and cosines of the half-sides and half-angles, which he called formulas of the second kind. Among the latter there are, for example, the elegant formulas first published by Delambre (1807) and later independently by Mollweide (1808) and by Gauss (1809). What is remarkable about the Delambre formulas is that they carry a double sign and that there exist two classes of spherical triangle, one for which the upper sign is valid and another for which the lower sign is. It was Study [4] who grasped the full significance of these formulas; he showed that to establish their full import it was necessary to extend the range of values of the sides and angles of a spherical triangle to include all real values. We intend in this paper to prove Delambre's formulas, and analogous ones for spherical

polygons, by means of simple applications of some elementary transformations of oriented lineal elements on a sphere, which, besides being intrinsically interesting, provide a procedure for obtaining more general results and, in addition, serve to make more natural Study's concept of a spherical triangle. The transformations we have in mind are special cases of one called the *spherical whirl* [1].

2. Turns and slides. Let the unit sphere S have its center at O, the origin of a right-handed orthogonal Cartesian frame $\mathfrak{F}_0$. Let an oriented lineal element* $\mathcal{E}$ be tangent to S at the point P; then we call the great circle passing through P tangent to $\mathcal{E}$ and oriented in the same sense as $\mathcal{E}$, the great cycle of $\mathcal{E}$. We assume all lineal elements are oriented. Let the lineal element $\mathcal{E}_0$ have its point at (1, 0, 0) and let it be directed so that its great cycle passes through (0, 1, 0) and has for its positive orientation the counterclockwise sense when viewed from the point (0, 0, 1). We shall call $\mathcal{E}_0$ the primitive lineal element on S and $\mathfrak{F}_0$, its associated frame.

Let e_0, e_1, e_2, e_3 be the Hamilton quaternion units such that $e_1^2=e_2^2=e_3^2=e_1e_2e_3=-1$, $e_0e_i=e_ie_0=e_i$, and let

$$x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3, \qquad \bar{x} = x_0e_0 - x_1e_1 - x_2e_2 - x_3e_3.$$

Then a rotation of S around an arbitrary diameter is given by the Hamilton-Cayley formula

$$N(x)u^* = \bar{x}ux, \qquad N(x) = x\bar{x},$$

where u and u^* are unit vectors (pure quaternions) with their initial points at O. The components x_i of x are the homogeneous Euler parameters of the rotation. Let $\mathcal{E}$ be an arbitrary lineal element on S', a unit sphere concentric with S, and let x be the quaternion of the rotation of S such that $\mathcal{E}_0 \to \mathcal{E}$; then we let, as we may, the components of x serve as the homogeneous coordinates of $\mathcal{E}$. For convenience, where no confusion will result, we shall let x designate both the rotation $\mathcal{E}_0 \to \mathcal{E}$ and $\mathcal{E}$. Evidently, if x designates $\mathcal{E}$, so does kx, where k is a nonzero scalar. If $N(x)=1$, x is said to be normalized. To any quaternion x, $N(x)\neq 0$, correspond two normalized ones, differing only in sign, namely $\pm x/[N(x)]^{1/2}$. We assume from here on that all quaternions are normalized.

If the rotation x revolves S around the unit vector v through the angle 2θ, we can set $x=-\cos\theta+v\sin\theta$, so that its associated normalized quaternion

$$-x = -\cos(\pi-\theta) + \bar{v}\sin(\pi-\theta), \qquad \bar{v} = -v,$$

designates a rotation of angle $2\pi-2\theta$ around $\bar{v}$. The rotation x also rotates $\mathfrak{F}_0$ into another Cartesian frame $\mathfrak{F}$ situated relative to $\mathcal{E}$ as $\mathfrak{F}_0$ is situated relative to $\mathcal{E}_0$; we shall call $\mathfrak{F}$ the frame associated with $\mathcal{E}$. The point (0, 0, 1) of $\mathfrak{F}$ will be called the pole of the great cycle of $\mathcal{E}$.

A lineal element transformation $x \to x^*$ given by

* A *lineal element* on a sphere S is the combination of a point P on S and a line t tangent to S at P. The lineal element is said to be *oriented* if t is oriented.

$$x^* = xa, \tag{2.1}$$

where $a = -\cos\alpha + u\sin\alpha$, $u^2 = -1$, represents a lineal element rotation by means of which all lineal elements on S are rotated through an angle 2α around an oriented diameter having the direction of the vector u. Now, let us suppose that every lineal element x is rotated through the same angle 2α, but in such a way that each lineal element revolves about an oriented axis situated relative to its associated frame $\mathfrak{F}$ as the axis u is situated relative to the primitive frame $\mathfrak{F}_0$. Such a transformation we have called a *spherical whirl* [1]. Evidently, if the whirl transports $x \to x^*$, $x^*\bar{x} = (x\bar{x})a$, so that the whirl has the equation

$$x^* = ax. \tag{2.2}$$

The whirls (2.2) constitute a three-parameter group $\mathfrak{W}_3$, skew-isomorphic to the group of lineal element rotations (2.1). We shall be particularly concerned with certain one-parameter subgroups of $\mathfrak{W}_3$, namely, the *turns*, the *slides* and the *dilatations*.

A *turn* T_τ is a whirl such that every lineal element is rotated around its point through an angle τ. The group of turns is given by

$$T_\tau\colon x^* = [-\cos(\tau/2) + e_1\sin(\tau/2)]x. \tag{2.3}$$

A *slide* S_σ is a whirl such that every lineal element slides tangentially along its great cycle through the arc-length σ. The equation of the group of slides is

$$S_\sigma\colon x^* = [-\cos(\sigma/2) + e_3\sin(\sigma/2)]x. \tag{2.4}$$

It will be observed that the identity turn and identity slide are $T_{2k\pi}$ and $S_{2k\pi}$, respectively, where k is an integer, and that the identity transformation is

$$I\colon x^* = \rho x,\ \rho = \pm 1, \tag{2.5}$$

ρ taking the upper sign when k is odd and the lower one when k is even.

A *dilatation* D_δ is a whirl such that every lineal element x is transported to a position orthogonal to a great cycle c which is itself orthogonal to the great cycle of x, the point of x traversing an arc of length δ on c. The dilatations form a one-parameter group. The sense of c is chosen so that the equation of this group is given by

$$D_\delta\colon x^* = [-\cos(\delta/2) + e_2\sin(\delta/2)]x. \tag{2.6}$$

The dilatations, unlike the turns and slides, are contact transformations. Evidently,

$$D_\delta = T_{-\pi/2}S_\delta T_{\pi/2} = S_{\pi/2}T_\delta S_{-\pi/2}$$

and

$$T_\alpha T_{-\alpha} = S_\sigma S_{-\sigma} = D_\delta D_{-\delta} = I.$$

We define a lineal element polarity $\mathcal{P}$ by means of the equation

$$\mathfrak{P}: x^* = T_\pi D_{\pi/2} x; \tag{2.7}$$

clearly, we can express $\mathfrak{P}$ also in the equivalent forms:

$$\mathfrak{P} = T_\pi D_{\pi/2} = T_{\pi/2} S_{\pi/2} T_{\pi/2} = D_{-\pi/2} T_\pi = S_{\pi/2} T_{\pi/2} S_{\pi/2}. \tag{2.8}$$

With the aid of (2.3) and (2.6) we obtain for $\mathfrak{P}$ the equation

$$\mathfrak{P}: x^* = (e_1 + e_3)2^{-1/2} x;$$

from this, or from (2.8), we get $\mathfrak{P}^2 x = -x$; that is, $\mathfrak{P}^2 = I$. Also by means of these equations we see that

$$S_\sigma = \mathfrak{P} T_\sigma \mathfrak{P}, \qquad T_\sigma = \mathfrak{P} S_\sigma \mathfrak{P}. \tag{2.9}$$

As the lineal element x slides along its great cycle c_1 a distance σ, its polar, $\mathfrak{P}x$, turns around its point P_1, which is the pole of c_1, through an angle σ; and, as x rotates around its point P_2 through an angle σ, its polar, $\mathfrak{P}x$, slides on its great cycle, of which P_2 is the pole, a distance σ.

Let x and y be two lineal elements on S. If x and y have a common point the whirl $x \to y$ is a turn; if x and y have a common great cycle the whirl $x \to y$ is a slide. If they have neither a common point nor a common great cycle, let P_1 and c_3 be the point and great cycle respectively of x; P_3 and c_1 the point and great cycle of y. Let P_2 be a point of intersection of c_3 and c_1, and let c_2 be a great cycle that passes through P_3 and P_1. Let us now subject x to a sequence of slides on the great cycles c_i and turns at the points P_i, as follows:

(1) a slide S_σ on c_3, starting from P_1 and terminating at P_2, the slide including possibly any number of circuits of c_3;

(2) a turn T_β at P_2, starting from the direction of c_3 and terminating in a direction properly tangent to c_1, permitting the mobile lineal element to complete any number of revolutions during the turn;

(3) a slide S_a from P_2 to P_3;

(4) a turn T_γ from c_1 to c_2;

(5) a slide S_b from P_3 to P_1; and

(6) a turn T_α from c_2 to c_3. With T_α, the lineal element is brought back to its initial position. The complete circuit is given by

$$T_\alpha S_b T_\gamma S_a T_\beta S_c x = \rho x, \qquad \rho = \pm 1. \tag{2.10}$$

3. Spherical polygons. We define a Study spherical triangle by means of a circuit (2.10); the slides determine its sides and the turns its angles. Degenerate triangles are not excluded. The first three steps in the circuit (2.10) produce the whirl $x \to y$. Multiplying the two members of (2.10) on the right by $\bar{x}$ and on the left by $T_{-\gamma} S_{-b} T_{-a}$, we obtain two equivalent forms for the whirl, namely,

$$S_a T_\beta S_c = \rho T_{-\gamma} S_{-b} T_{-\alpha}. \tag{3.1}$$

By means of great cycles passing through P_1 and P_3 and orthogonal to c_1 and c_3, respectively, the whirl (3.1) can be resolved also into products such as TDS,

SDT, etc., where T, D and S represent suitable turns, dilatations and slides.

Since any quaternion defines a whirl, it can be expressed in any one of these factored forms. Consequently, if we replace x in the rotation formula $u^* = \bar{x}ux$ by one of the sets of three factors into which it can be resolved, we obtain for each set a parametric representation of the rotation, the parameters representing the magnitudes of the slides, turns and dilatations that occur in the product. In particular, we get Euler's parametric form when we let $x = S_a T_\beta S_c$.

Let x be a rotation of angle θ around an axis carrying the unit vector $v = e_1 \cos\lambda + e_2 \cos\mu + e_3 \cos\nu$; then we can let

$$x \equiv -\cos(\theta/2) + v \sin(\theta/2) = T_a S_\beta S_c.$$

Substituting for the turns and slides from (2.3) and (2.4), we obtain, by equating the corresponding components:

$$\begin{aligned}
\cos\frac{\theta}{2} &= \cos\frac{\beta}{2}\cos\frac{a+c}{2}, \\
\cos\lambda \sin\frac{\theta}{2} &= \sin\frac{\beta}{2}\cos\frac{a-c}{2}, \\
\cos\mu \sin\frac{\theta}{2} &= \sin\frac{\beta}{2}\sin\frac{a-c}{2}, \\
\cos\nu \sin\frac{\theta}{2} &= \cos\frac{\beta}{2}\sin\frac{a+c}{2}.
\end{aligned} \tag{3.2}$$

If we multiply both members of (2.10) on the left by $\mathcal{P}$ and insert $\mathcal{P}\mathcal{P} = I$ between the factors in the left-hand member, we obtain:

$$(\mathcal{P}T_a\mathcal{P})(\mathcal{P}S_b\mathcal{P})(\mathcal{P}T_\gamma\mathcal{P})(\mathcal{P}S_a\mathcal{P})(\mathcal{P}T_\beta\mathcal{P})(\mathcal{P}S_c\mathcal{P})\mathcal{P}x = \rho\mathcal{P}x;$$

hence, by virtue of (2.9), we see that as x makes a circuit of the Study triangle (2.10), its polar, $\mathcal{P}x$, makes a circuit of its polar triangle.

Inserting in (3.1) the formulas for the turns and slides given by (2.3) and (2.4), we obtain the formulas of Delambre:

$$\begin{aligned}
\cos\frac{\beta}{2}\cos\frac{a+c}{2} &= -\rho\cos\frac{b}{2}\cos\frac{\alpha+\gamma}{2}, \\
\sin\frac{\beta}{2}\cos\frac{a-c}{2} &= \rho\cos\frac{b}{2}\sin\frac{\alpha+\gamma}{2}, \\
\sin\frac{\beta}{2}\sin\frac{a-c}{2} &= -\rho\sin\frac{b}{2}\sin\frac{\alpha-\gamma}{2}, \\
\cos\frac{\beta}{2}\sin\frac{a+c}{2} &= \rho\sin\frac{b}{2}\cos\frac{\alpha-\gamma}{2},
\end{aligned} \qquad (\rho = \pm 1). \tag{3.3}$$

It will be observed that the left members of the equations (3.3) appear also in equations (3.2); consequently, with every Study triangle there is associated an orthogonal transformation, the axis and angle of which are given by equations (3.2). By cyclic permutation we get two more sets of equations like (3.3).

There are Study triangles for which only those Delambre formulas are valid in which $\rho = +1$, and also others for which the Delambre formulas with $\rho = -1$ are valid. The full significance of the double sign was made clear by Study [4, 5]. In order to formulate Study's principal theorem bearing on this circumstance, let us, with Study, call the triangles belonging to the class in which $\rho = +1$ *proper*, and those belonging to the class in which $\rho = -1$ *improper*. Furthermore, let us consider the six-space S_6 the points of which have for their coordinates the quantities $a, b, c, \alpha, \beta, \gamma$, all of which are unrestricted real numbers. Then Study's theorem can be stated as follows:

The set of points in S_6 which are the images of the aggregate of Study triangles separates into just two continua (compact, connected point sets), one corresponding to the set of proper triangles and the other to the set of improper triangles.

The Euler triangles belong to the set of proper triangles.

In the same manner we can, of course, by means of a circuit of slides and turns, define a spherical polygon of any number of sides; the slides determine the sides in sense and magnitude while the turns, in like manner, determine the angles. The polarity (2.7) assigns to every polygon a unique polar polygon. The sides and angles are unrestricted in magnitude, as in the case of Study triangles. Starting from the equation for a circuit of slides and turns for such a polygon, like equation (2.10) for triangles, we can derive sets of formulas connecting its sides and angles, analogous to those of Delambre. Let it suffice to indicate the results obtainable for a spherical quadrilateral determined by the following sequence of slides and turns:

$$T_\beta S_d T_\gamma S_c T_\beta S_b T_\alpha S_a x = \rho x, \qquad \rho = \pm 1. \tag{3.4}$$

Multiplying both of the members of (3.4) on the right by $\bar{x}$, and on the left by $S_{-c}T_{-\gamma}S_{-d}T_{-\beta}$, we obtain

$$T_\beta S_b T_\alpha S_a = \rho S_{-c} T_{-\gamma} S_{-d} T_{-\beta}. \tag{3.5}$$

Substituting in (3.5) the formulas for the slides and turns given by (2.3) and (2.4), and simplifying the results, we obtain the following four formulas:

$$\left[\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha-\beta}{2}\right]\cos\frac{a+b}{2} + \left[\cos\frac{\alpha+\beta}{2} - \cos\frac{\alpha-\beta}{2}\right]\cos\frac{a-b}{2}$$
$$= \rho\left\{\left[\cos\frac{\gamma+\delta}{2} + \cos\frac{\gamma-\delta}{2}\right]\cos\frac{c+d}{2} + \left[\cos\frac{\gamma+\delta}{2} - \cos\frac{\gamma-\delta}{2}\right]\cos\frac{c-d}{2}\right\},$$

$$\left[\sin\frac{\alpha+\beta}{2}+\sin\frac{\alpha-\beta}{2}\right]\cos\frac{a-b}{2}+\left[\sin\frac{\alpha+\beta}{2}-\sin\frac{\alpha-\beta}{2}\right]\cos\frac{\alpha+b}{2}$$
$$=-\rho\left\{\left[\sin\frac{\gamma+\delta}{2}+\sin\frac{\gamma-\delta}{2}\right]\cos\frac{c-d}{2}+\left[\sin\frac{\gamma+\delta}{2}-\sin\frac{\gamma-\delta}{2}\right]\cos\frac{c+d}{2}\right\},$$

$$\left[\sin\frac{\alpha+\beta}{2}+\sin\frac{\alpha-\beta}{2}\right]\sin\frac{a-b}{2}+\left[\sin\frac{\alpha+\beta}{2}-\sin\frac{\alpha-\beta}{2}\right]\sin\frac{a+b}{2}$$
$$=-\rho\left\{\left[\sin\frac{\gamma+\delta}{2}+\sin\frac{\gamma-\delta}{2}\right]\sin\frac{c-d}{2}+\left[\sin\frac{\gamma+\delta}{2}-\sin\frac{\gamma-\delta}{2}\right]\sin\frac{c+d}{2}\right\},$$

$$\left[\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha-\beta}{2}\right]\sin\frac{a+b}{2}+\left[\cos\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}\right]\sin\frac{a-b}{2}$$
$$=-\rho\left\{\left[\cos\frac{\gamma+\delta}{2}+\cos\frac{\gamma-\delta}{2}\right]\sin\frac{c+d}{2}+\left[\cos\frac{\gamma+\delta}{2}-\cos\frac{\gamma-\delta}{2}\right]\sin\frac{c-d}{2}\right\}.$$

Cyclic permutations of the sides and angles provide three more sets of formulas. It will be observed that these formulas are self-polar; moreover, as in the case of Study triangles, there are two kinds of quadrilaterals: those whose sides and angles satisfy the formulas with $\rho=+1$, and those that satisfy the set with $\rho=-1$.

In [1] we defined a *spherical turbine* as a series of oriented lineal elements, the points of which lie on a cycle c, and the great cycles of which make a constant angle α with c. When $\alpha=0$ we have a *tangential turbine*, and, when c is a point cycle, a *point turbine*. The equation in lineal element coordinates of the point turbine at the point e_1 is clearly $\bar{x}e_1x=e_1$. By means of a lineal element rotation (2.1), this is transformed into a point turbine at an arbitrary point having for its cordinates the components of the unit vector r, namely, $\bar{x}e_1x=r$. The dilatation D_δ transforms this point turbine into a tangential one with center at r and radius equal to δ, given by the equation

$$\bar{x}(e_1\cos\delta-e_3\sin\delta)x=r. \tag{3.6}$$

When $\delta=\pi/2$, (3.6) is the equation of a great cycle.

If we let the four components x_i of the quaternion x be the homogeneous coordinates of a point in projective three-space S_3, we obtain a continuous one-

to-one mapping of the lineal elements x on S upon the points x of S_3. By virtue of this mapping, the ∞^6 lineal element *whirl-rotations* $x \to axb$, are mapped on the ∞^6 displacements in elliptic three-space E_3; indeed, to the whirls $x \to ax$ correspond the left translations in E_3, and to the rotations $x \to xb$ correspond the right translations [1]. To the turbines on S correspond one-to-one the lines in E_3. To a slide of the lineal elements on S corresponds a left translation in which the points of E_3 glide on lines belonging to a congruence of Clifford parallels C_1, and to a turn on S corresponds a left translation on the lines of another congruence of Clifford parallels C_2. The sequence of slides and turns in (2.10), defining a triangle on S, corresponds to a sequence of left translations that a point in E_3 executes in describing a complete circuit of a skew hexagon, the sides of which belong alternately to C_1 and C_2. Similarly, to a spherical polygon of n sides defined by slides and turns, there corresponds in E_3 a polygon of $2n$ sides that lie alternately on lines belonging to C_1 and C_2. If, however, we let a point in E_3 make a circuit, by means of left translations, of a polygon, say a hexagon, the sides of which are not restricted to C_1 and C_2, the corresponding lineal element on S executes a sequence of six whirls in each of which an "arc" of a turbine is traversed; the figure on S that corresponds to a general hexagon in E_3 is consequently a closed chain of six "arcs" of turbines. For such a figure we could also obtain formulas analogous to those of Delambre; however, the lengths of the "arcs" would be given in an elliptic metric and the formulas would involve additional parameters, namely, the six radii of the turbines and the angles of inclination of the lineal elements in the turbines.

An interesting special case arises when the six turbines are alternately tangential turbines with radii between 0 and $\pi/2$, and point turbines. Such a figure is a "spherical triangle" t whose sides are oriented arcs of small cycles. Let us assume that the three small cycles have equal radii, and that the sides and angles of t are measured in the elliptic metric of their image in E_3. It can be easily verified that the tangential whirl that makes all of the lineal elements of the cycle (3.6) advance along its periphery for the distance $a/2$ is given by

$$(3.7) \qquad W_a\colon x \to qx, \qquad q = -\cos(a/2) + (e_1 \cos\delta - e_3 \sin\delta)\sin(a/2).$$

It should be observed that

$$W_a = D_{-\delta} T_a D_\delta, \qquad W_{-a} = D_{\delta'} S_a D_{-\delta'}, \qquad \delta' = \pi/2 - \delta.$$

We can now define the sides and angles of t by means of the following circuit:

$$(3.8) \qquad T_\beta W_c T_a W_b T_\gamma W_a x = \rho x, \qquad \rho = \pm 1,$$

where $a/2$, $b/2$, $c/2$, are the lengths of the sides of t, and $\alpha/2$, $\beta/2$, $\gamma/2$, are the magnitudes of its angles. From (3.8) we obtain

$$(3.9) \qquad W_b T_\gamma W_a = \rho T_{-a} W_{-c} T_{-\beta}.$$

Substituting for the turns and tangential whirls in (3.9) the expressions given by (2.3) and (3.7), we obtain the following formulas:

$$\cos\delta \sin\frac{\gamma}{2}\sin\frac{a+b}{2} - \cos\frac{\gamma}{2}\cos\frac{a+b}{2}$$

$$= \rho\left[\cos\delta \sin c \sin\frac{\alpha+\beta}{2} - \cos\frac{c}{2}\cos\frac{\alpha+\beta}{2}\right],$$

$$\cos\delta\cos\frac{\gamma}{2}\sin\frac{a+b}{2} - \cos 2\delta \sin\frac{\gamma}{2}\sin\frac{a}{2}\sin\frac{b}{2} + \sin\frac{\gamma}{2}\cos\frac{a}{2}\cos\frac{b}{2}$$

$$= -\rho\left[\cos\delta \sin\frac{c}{2}\cos\frac{\alpha+\beta}{2} + \cos\frac{c}{2}\sin\frac{\alpha+\beta}{2}\right],$$

$$\sin 2\delta \sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{\gamma}{2} - \sin\delta \sin\frac{a+b}{2}\cos\frac{\gamma}{2} = \rho \sin\delta \cos\frac{\alpha-\beta}{2}\sin\frac{c}{2}.$$

When $\delta=\pi/2$, t becomes a Study triangle, and these three formulas reduce to Delambre's. However, since the sides and angles are measured in an elliptic metric in which π is the magnitude of one complete revolution of x around its point, as well as of a complete circuit by x of its great cycle, we accordingly let a, b, c be the lengths of the sides and α, β, γ the magnitudes of the angles, to restore the spherical metric.

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AN INEQUALITY ARISING IN GENETICAL THEORY

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1. The study of genetics provides simple examples of the use of such mathematical tools as probability theory, calculus, and matrices: and it may well be of interest to teachers of mathematics and statistics. Recently one of us (C. A. B. S.) came across an unusual inequality in a genetical problem, and the other author (H. P. M.) succeeded in proving its correctness. The inequality is as follows:

Let v, u be arbitrary nonzero real $k\times 1$ and $k\times k$ matrices respectively, with all their elements nonnegative, and with u symmetrical, i.e., $u^T=u$, where u^T denotes the transpose of u. Then, for any positive integer n,

$$(1) \qquad (v^T u^n v)(v^T v)^{n-1} \geqq (v^T u v^n)$$

with equality if and only if v is a latent vector of u.

We have received alternative proofs of the first part of this result from M. M. Crum and R. Scheuer, and a weaker result has been published by S. P. M. Mandel and I. M. Hughes [4] since this paper was first written. We explain here why this inequality is of interest, as well as giving a proof.

2. One of the most striking of biological phenomena is the great variability of many species of plants and animals, even when raised in a nearly constant environment. For example, some men are short, others tall, some are clever, others stupid, and some dark, others fair. It is far from being immediately obvious why this should be so. For if natural selection leads to the survival of the fittest, why has humanity not been reduced to a single uniform type of maximum fitness? It would be very interesting to know the reason; but it would also be very presumptuous at present to attempt a detailed explanation for qualities such as height, intelligence, or hair color, since our knowledge of how these are inherited is very incomplete. Instead, we here consider a rather simplified model. In fact, in what follows both the biological facts and the theoretical model have been, strictly speaking, oversimplified, but with the essential features retained. Those who wish to pursue the matter further can read C. C. Li [3] or O. Kempthorne [2], who give references to other work on the subject.

Many properties of living organisms are determined by objects called *genes*. For example, there are three types of genes, usually called A, B and O, that determine the ordinary blood group in man; each individual carries just two such genes, so that there are 6 distinct blood groups, or pairs of genes, namely OO, AO, AA, BO, BB, AB. Their practical importance is that it is dangerous to transfuse blood from an individual carrying an A gene into one without an A gene, *e.g.*, AO or AA blood may not be transfused into a BO or BB patient, and a similar rule holds for the B gene. (It can be deduced that when only transfusion is considered it is unnecessary to distinguish between the types AO and AA, which are lumped together as *group A*; similarly $BO+BB=$*group B*.) Mendel's laws state that each child gets at random one of the two genes carried by its father, and independently and at random one of the two carried by its mother. Thus, if the father is AB and the mother OO the child is equally likely to get A or B but must get an O; it is therefore equally likely to be AO or BO. A child of an AB father and AO mother is equally likely to be AA, AO, AB, or BO.

In general we consider a set of k types of genes or *alleles*, $G_1, \cdots, G_k$, with the property that any individual carries just two such genes, possibly of the same type. In a large population of N individuals there will be a certain number

N_{rs} of type G_rG_s: the ratio $N_{rs}/N = p_{rs}$ is the frequency of type G_rG_s in the population. The frequencies p_{rs} describe the population as far as the genes G_r are concerned. Thus in London approximately 7 per cent of persons have blood group AA, 35 per cent AO, 0.4 per cent BB, 8 per cent BO, 3 per cent AB, and 47 per cent OO (*blood group O*). The population of N individuals can also be considered as one of $2N$ genes. Thus, since every G_1G_1 individual carries two G_1 genes, and every G_1G_r ($r \neq 1$) carries one, the number of G_1 genes is

$$N_1 = 2N_{11} + N_{12} + \cdots + N_{1k}.$$

A division by $2N$, the total number of genes, shows that the frequency of G_1 is

$$f_1 = N_1/2N = p_{11} + \tfrac{1}{2}p_{12} + \cdots + \tfrac{1}{2}p_{1k}, \tag{2}$$

and similarly for $f_2, \cdots, f_k$. Thus in London the A gene has a frequency $7+\frac{1}{2}\cdot 35+\frac{1}{2}\cdot 3 = 26$ per cent. In a large population it is generally found that the frequencies are the same in males and females, and so equality is assumed in the following argument.

Evolutionary genetics is the study of how a population gradually changes from generation to generation. For simplicity we here assume the generations to be separate, as with annual plants, which have one generation each year. $p_{rs}(g), f_r(g)$ represent the frequencies in the gth generation. Now there is a saying that "marriage is a lottery." If this is taken to mean that marriage partnerships are formed at random, it seems to be true as far as blood groups and similar qualities are concerned. For while one may choose a partner for charm, intelligence, beauty or common interests, it isn't likely that his or her blood group will seem particularly adorable. We make the assumption that the genes under consideration do not affect the chances of having children: this means that a baby chosen at random gets at random one of the genes carried by a random father, and similarly and independently a random gene via the mother. Thus the chance of a baby of generation $(g+1)$ being of type G_rG_r is $f_r(g)\cdot f_r(g)$, while the chance of being G_rG_s ($r \neq s$) is

$$f_r(g)\cdot f_s(g) + f_s(g)\cdot f_r(g) = 2f_r(g)\cdot f_s(g)$$

since either the G_r gene comes from the father and the G_s from the mother or vice versa. But babies with different genes will have different liabilities to disease or accident: we let w_{rs} be the chance of a baby of type G_rG_s surviving to adult life. The matrix $\mathbf{w}$ of the w_{rs} accordingly represents the effect of natural selection. The frequencies $p_{rs}(g+1)$ of the various types G_rG_s among adults of generation $(g+1)$ will be in the ratios of the frequencies at birth multiplied by the survival probabilities, *i.e.*, for the types G_1G_1, G_1G_2, G_1G_3, $\cdots$ they will be in the ratios

$$\begin{aligned} &w_{11}[f_1(g)]^2 : 2w_{12}f_1(g)f_2(g) : 2w_{13}f_1(g)f_3(g) : \cdots \\ &\qquad = p_{11}(g+1) : p_{12}(g+1) : p_{13}(g+1) : \cdots . \end{aligned} \tag{3}$$

To simplify the notation we introduce here certain conventions. Unless otherwise stated p_{rs}, f_r stand for the frequencies $p_{rs}(g), f_r(g)$ in the gth generation, and P_{rs}, F_r for the corresponding values in the following, $(g+1)$th, generation. After a summation sign Greek letters will be used to denote the suffixes to be summed over, and these will be allowed to range over all possible values unless the contrary is indicated: for example, $\sum f_\alpha = 1$, and (2) becomes $f_r = \frac{1}{2}p_{rr} + \frac{1}{2}\sum p_{r\alpha}$.

Now it may happen trivially that all the products $w_{rs}f_rf_s$ are zero, so that there are no survivors in generation $g+1$; we suppose this case excluded. Since the frequencies $P_{rs} = p_{rs}(g+1)$ must add up to unity, and their ratios are given by (3), their actual values are

$$P_{rr} = w_{rr}f_r^2/W(f), \qquad P_{rs} = 2w_{rs}f_rf_s/W(f) \qquad (r \neq s),$$

where $W(f)$ stands for the quadratic form $f^T w f = \sum w_{\alpha\beta}f_\alpha f_\beta > 0$. From (2) the gene frequencies in generation $g+1$ are

$$F_r = f_r \sum w_{r\alpha}f_\alpha/W(f) = \tfrac{1}{2}f_r[W(f)]^{-1}\partial W(f)/\partial f_r, \tag{4}$$

which we write for brevity as $F = \phi(f)$. This is a recurrence relation connecting the gene frequencies in generations g and $g+1$. It is easily seen that if $w_{rs}f_rf_s > 0$ then $w_{rs}F_rF_s > 0$ so that, by assuming that the products $w_{rs}f_rf_s$ do not all vanish, we have ensured that the population will continue to exist in generation $g+2$, and the frequencies in that generation are obtained from those of generation $g+1$ by equations similar to (4). By induction, the population continues to exist in every subsequent generation $g^* > g$, and $W(f(g^*)) > 0$, and a relation like (4) connects every two successive generations.

A real numerical example is provided by Allison [1]. In central Africa there are genes affecting the hemoglobin (officially called Hb^A, Hb^S, but here we use the letters G_1, G_2, for consistency) such that G_1G_1 type individuals (the normal European type) are susceptible to malaria, while the G_2G_2 tend to die in childhood from a severe sickle-cell type of anemia. Their survival chances are, according to Allison, approximately in the ratio $w_{11}:w_{12}:w_{22} = 0.8:1:0.2$, so that the recurrence relation is, by (4),

$$F_1 = f_1(w_{11}f_1 + w_{12}f_2)/(w_{11}f_1^2 + 2w_{12}f_1f_2 + w_{22}f_2^2), \tag{5}$$

where $f_2 = 1 - f_1$. Thus if we take, say, $f_1(1) = .1$ as initial value, the frequency of G_1 in successive generations will be $f_1(2) = .280$, $f_1(3) = .464$, $f_1(4) = .579$, $\cdots$ tending eventually to .8, whereas if initially $f_1(1) = .9$ the successive values are .889, .878, .868, etc., again tending to the limit .8. From this it seems plausible that whatever frequencies (other than 0 or 1) the genes G_1 and G_2 have initially, they will approximate to $f_1 = .8$, $f_2 = .2$ respectively after a few generations; $f_1 = .8$ is a *stable equilibrium*. Thus all three types G_1G_1, G_1G_2, G_2G_2 will continue to exist indefinitely in the population *as a result of natural selection*. Of course, if initially $f_1 = 0$, it will always remain so, for in our model if a gene is not present in any one generation it cannot appear in any subsequent generation. (In nature

it could be introduced by migration or mutation.) We call an equilibrium *degenerate* if any of the f_r are there zero, otherwise *nondegenerate.*

An example of a stable equilibrium with three genes G_1, G_2, G_3 is shown in Figure 1. Here the selection matrix is taken to be

$$w = \begin{bmatrix} .4 & .7 & .6 \\ .7 & .1 & .9 \\ .6 & .9 & .4 \end{bmatrix}.$$

The small circles show the values of the three gene frequencies f_1, f_2, f_3 plotted as trilinear coordinates, which is possible since $f_1+f_2+f_3=1$, and the initial values are various points near the perimeter of the triangle. The sequences of points clearly appear to tend to an equilibrium at $E=(\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$.

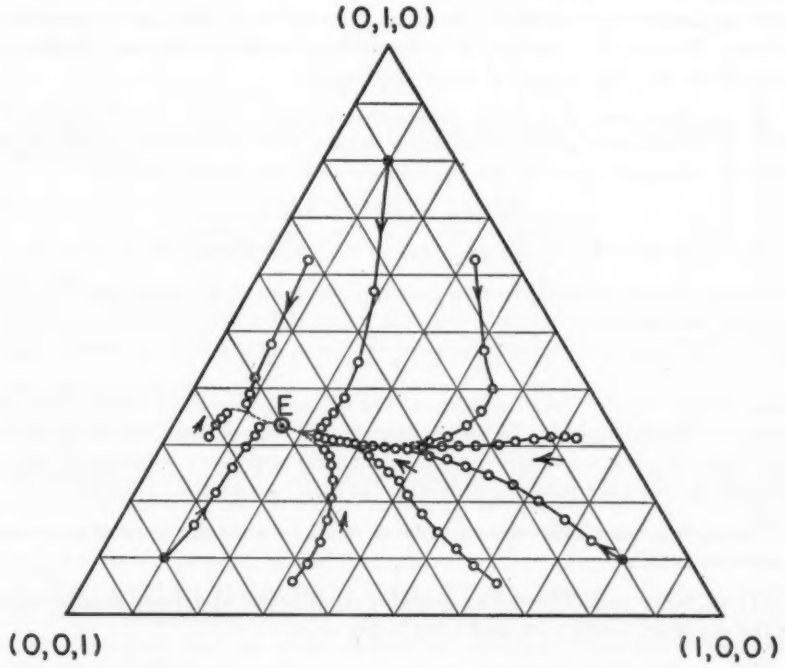

FIG. 1

3. It is therefore an important genetical problem to know when such stable equilibria exist, and whether a population not initially in equilibrium must tend to one under the action of selection.

When only two genes G_1 and G_2 are concerned, a complete analysis can be

made by elementary methods. The cases in which $w_{11}w_{22}=0=w_{12}$ are trivial in the sense that at least one gene is completely lost after the first generation. We therefore exclude these, as well as the cases for which $w_{11}f_1^2=w_{12}f_2f_2^2=w_{22}f_2=0$, already excluded in Section 2. For other cases the analysis is most easily done in terms of the ratio $r=f_1/f_2=f_1/(1-f_1)$, which has the range of values $0\leqq r\leqq\infty$; if $R=F_1/F_2$ the recurrence relation (5) becomes

$$R=r(w_{11}r+w_{12})/(w_{12}r+w_{22}) \tag{6}$$

(and $R=\infty$ when $r=\infty$). An equilibrium point is one at which $R=r$. From (6) we see that there are always two degenerate equilibria $r=0$ and ∞ (at which only one gene is present); there is also a third, nondegenerate equilibrium $r=q$ where

$$q=(w_{12}-w_{22})/(w_{12}-w_{11}) \tag{7}$$

provided that $0<q<\infty$, unless we have $w_{11}=w_{12}=w_{22}$, and q is indeterminate: but then $R=r$ for all r, and any population is in a state of (neutral) equilibrium. Apart from this, the following cases may occur:

(I) $w_{11}<w_{12}>w_{22}$. A nondegenerate equilibrium q exists. Furthermore, it is stable, and whatever initial value other than 0 or ∞ we take for r, the population will eventually tend to the equilibrium q. For from (6) and (7)

$$\frac{R-q}{r-q}=\frac{w_{11}r+w_{22}}{w_{12}r+w_{22}}=1-\frac{(w_{12}-w_{11})r}{w_{12}r+w_{22}}.$$

Thus if we choose arbitrarily two positive numbers r', r'' such that $r'<r<r''$, $r'<q<r''$, we have

$$0<(R-q)/(r-q)<1-c, \tag{8}$$

where $c=(w_{12}-w_{11})r'/(w_{12}r''+w_{22})>0$. The distance $|r-q|$ of r from the equilibrium q is diminished by a factor smaller than $1-c$ in one generation. It follows that $r'<R<r''$, the argument repeats in the next and every subsequent generation, and the distance tends to zero with (at least) geometric rapidity.

The other cases which follow can be covered by a similar type of argument, so we omit details:

(II) $w_{11}>w_{12}<w_{22}$. There is an unstable equilibrium at q; from smaller values of r the sequence tends to 0, and from larger ones, to ∞.

(III) $w_{11}\leqq w_{12}\leqq w_{22}$ (but not $w_{11}=w_{12}=w_{22}$). For any initial r, except ∞, successive values tend to 0, the only stable equilibrium.

(IV) $w_{11}\geqq w_{12}\geqq w_{22}$ (but not $w_{11}=w_{12}=w_{22}$). As case (III), with 0 and ∞ interchanged.

4. In general, if $f(1)$ is the initial gene-frequency vector, the vectors in subsequent generations are $f(2)=\phi(f(1))$, $f(3)=\phi(f(2))$, and so on, where the func-

tion ϕ is defined explicitly by (4); but it is difficult to get any explicit expression for $f(i)$ for general i, or study the properties of the sequence directly. However, much information can be got from the following two theorems. (From now on, it is to be always understood that a frequency vector, such as $\boldsymbol{f}$, is to be restricted to lie in the simplex [all $f_i \geqq 0$, $\sum f_\alpha = 1$] and in the region where $W(\boldsymbol{f}) > 0$, unless the contrary is explicitly stated.)

THEOREM 1. *$\boldsymbol{E}$ is an equilibrium point if and only if either (trivially) only one $E_i \neq 0$ or the function $W(\boldsymbol{f})$ has a stationary point at $\boldsymbol{f} = \boldsymbol{E}$ when variation is allowed only in the components f_i for those i for which $E_i \neq 0$; if $E_j = 0$, f_j is held fixed at 0. In particular, a nondegenerate equilibrium $\boldsymbol{E}$ (i.e., one with all $E_i > 0$), is a stationary point for $W(\boldsymbol{f})$ with the f_i varying freely (apart from the necessary restriction $\sum f_\alpha = 1$).*

Proof. Suppose, for example, that $E_1 = E_2 = 0$, $E_i > 0$ $(i > 2)$. The necessary and sufficient conditions for $W(\boldsymbol{f})$ to be stationary at $\boldsymbol{E}$ subject to the conditions $f_1 = f_2 = 0$, $\sum f_\alpha = 1$ are that for some λ

$$\partial W(\boldsymbol{f})/\partial f_3 = \partial W(\boldsymbol{f})/\partial f_4 = \cdots = \partial W(\boldsymbol{f})/\partial f_k = \lambda \tag{9}$$

when $\boldsymbol{f} = \boldsymbol{E}$. By Euler's Theorem $\sum f_\alpha \partial W(\boldsymbol{f})/\partial f_\alpha = 2W(\boldsymbol{f})$, *i.e.*, $\lambda = \frac{1}{2}W(\boldsymbol{E})$. Thus (9) is equivalent to

$$f_i = \tfrac{1}{2} f_i [W(\boldsymbol{f})]^{-1} \partial W(\boldsymbol{f})/\partial f_i \tag{10}$$

for all i, when $\boldsymbol{f} = \boldsymbol{E}$: by (4) this means that $\phi(\boldsymbol{E}) = \boldsymbol{E}$, *i.e.*, $\boldsymbol{E}$ is by definition an equilibrium.

COROLLARY. *An equilibrium point $\boldsymbol{E}$ is either isolated or is on at least one straight segment on which every point $\boldsymbol{f}$ is an equilibrium point with $W(\boldsymbol{f}) = W(\boldsymbol{E})$.*

Let us consider, for example, those points $\boldsymbol{f}$ for which

$$f_1 = f_2 = 0, \qquad \partial W/\partial f_3 = \cdots = \partial W/\partial f_k, \qquad \sum f_\alpha = 1,$$

where, for the moment, we do not require every f_i to be nonnegative. Since these equations are linear in the components f_i they are satisfied either at no point, or at one point, or at every point of some r-dimensional linear subspace with $1 \leqq r \leqq k-3$. As in the proof of the theorem, on such a subspace $W(\boldsymbol{f})$ is stationary at every point for variation of $\boldsymbol{f}$ within the subspace, and thus $W(\boldsymbol{f})$ must clearly be constant on the subspace. On imposing the restrictions $f_i \geqq 0$ and using the theorem we see that the set of equilibrium points $\boldsymbol{E}$ of the special type with $f_1 = f_2 = 0$ is either empty, or contains only one point, or consists of an r-dimensional convex body (with $1 \leqq r \leqq k-3$) on which $W(\boldsymbol{f})$ is constant. Similarly for equilibrium points of the other special types. Since the number of possible types is finite (actually $2^k - 1$), it is clear that an equilibrium point must either be isolated or belong to a convex body (corresponding to one of the special types) in which every point is an equilibrium point and $W(\boldsymbol{f})$ is constant. This suffices to establish the assertion in the corollary.

THEOREM 2. $W(F) = W(\phi(f)) \geqq W(f)$, *with equality if and only if* f *is an equilibrium.*

(We recall that values of f at which $W(f) = 0$ have been explicitly excluded: they make F indeterminate.)

Proof. (We assume the correctness of (1), which will be proved later.) Write $v_i = \sqrt{f_i}$ and $u_{ij} = v_i w_{ij} v_j$. Then $W(f) = \sum w_{\alpha\beta} f_\alpha f_\beta = v^T u v$ and, from (4),

$$W(F) = \sum [w_{\alpha\beta} f_\alpha \sum w_{\alpha\delta} f_\delta \cdot f_\beta \sum w_{\beta\epsilon} f_\epsilon]/[W(f)]^2$$
$$= \sum v_\delta u_{\delta\alpha} u_{\alpha\beta} u_{\beta\epsilon} v_\epsilon / (v^T u v)^2 = (v^T u^2 v)/(v^T u v)^2.$$

Since $v^T v = \sum v_\alpha^2 = \sum f_\alpha = 1$, the inequality $W(F) \geqq W(f)$ is equivalent to $(v^T u^2 v)(v^T v)^2 \geqq (v^T u v)^2$. Furthermore from (9) and (10) the condition for equilibrium is that for each i either $f_i = v_i^2 = 0$ or $2\sum w_{i\alpha} f_\alpha = \lambda$, that is, for every i,

$$2 \sum v_i w_{i\alpha} v_\alpha^2 = \lambda v_i, \quad \text{or} \quad uv = \tfrac{1}{2}\lambda v,$$

i.e., that v is a latent vector of u. This reduces Theorem 2 to the case $n = 3$ of (1).

5. Now Theorem 2 implies that

$$W(f(1)) \leqq W(f(2)) \leqq W(f(3)) \leqq \cdots. \tag{11}$$

If the function $W(f)$ is represented by its graph, a "surface" in k dimensions, then the sequence $\{f(r)\}$ is represented by a sequence of points climbing up the surface. It is natural to suppose that they will eventually tend to a "peak," which will be a point of stable equilibrium. This picture of the genetic behavior of populations is due to Sewall Wright [5], but his formal proofs do not seem to be complete. In any case, unless we impose further restrictions upon the function $W(f)$ the situation may be complicated by the presence on the surface of saddle points, and also of level ridges or plateaus: the sequence of points may converge to a saddle point, which will be a point of unstable equilibrium, and (although we conjecture the contrary) it might have several limit points instead of converging to a unique limit. However, with the aid of Theorems 1 and 2, we are able to demonstrate the following theorem, and we shall leave till Section 6 the discussion of various genetically appropriate restrictions.

THEOREM 3. (I) *Any limit point of the sequence* $\{f(r)\}$ *is an equilibrium point.* (II) *Let* $W(f)$ *have a strict maximum at* E *(for admissible values of* f*): then* E *is a stable equilibrium point, i.e., if any point* $f(i)$ *is near enough to* E *then* $f(r) \to E$ *as* $r \to \infty$. (III) *If* E *is an isolated equilibrium point, then either* $f(r) \to E$ *as* $r \to \infty$ *or* E *is not among the limit points of* $\{f(r)\}$.

Proof. Since the sequence $\{W(f(r))\}$ is bounded and, by (11), is monotone, it must have a unique limit W_∞, say, as $r \to \infty$, and $W_\infty \geqq W(f(1)) > 0$. Also, $f(r)$ lies in the bounded closed region

$$R: \ \sum f_a = 1, \qquad \min f_a \geqq 0, \qquad W(\boldsymbol{f}) \geqq W(\boldsymbol{f}(1)).$$

Hence the sequence $\{\boldsymbol{f}(r)\}$ must have at least one limit point $\boldsymbol{L}$ in R: for any such $\boldsymbol{L}$ we have $W(\boldsymbol{L}) = W_\infty > 0$, since $W(\boldsymbol{f})$ is a continuous function of $\boldsymbol{f}$. Since $W(\boldsymbol{L}) > 0$ we have $\boldsymbol{\phi}(\boldsymbol{f})$ continuous at $\boldsymbol{L}$. Hence $\{\boldsymbol{\phi}(\boldsymbol{f}(r))\}$ must have $\boldsymbol{\phi}(\boldsymbol{L})$ as a limit point. But this sequence is $\{\boldsymbol{f}(r+1)\}$, and so $\boldsymbol{\phi}(\boldsymbol{L})$ is a limit point of $\{\boldsymbol{f}(r)\}$. Hence, as before, $W(\boldsymbol{\phi}(\boldsymbol{L})) = W_\infty$. Thus $W(\boldsymbol{\phi}(\boldsymbol{L})) = W(\boldsymbol{L})$; whence, by Theorem 2, $\boldsymbol{L}$ is an equilibrium point. We have now proved (I), and also that $W(\boldsymbol{L})$ has the same value for every limit point $\boldsymbol{L}$ of $\{\boldsymbol{f}(r)\}$.

For Part II, since $W(\boldsymbol{E})$ must clearly be positive, we can surround $\boldsymbol{E}$ by two regions S_1, S_2 consisting of the points $\boldsymbol{f}$ in R at which the distance d of $\boldsymbol{f}$ from $\boldsymbol{E}$ satisfies the respective inequalities $d \leqq \rho_1$, $d < \rho_2$, where ρ_1 and ρ_2 are such that $\rho_1 > \rho_2 > 0$ but otherwise are at our disposal. Now $W(\boldsymbol{f})$ has a strict maximum at $\boldsymbol{E}$, and so, by Theorem 1, Corollary, $\boldsymbol{E}$ is an isolated equilibrium point. Hence we can choose ρ_1 to be such that

(i) if $\boldsymbol{f}$ is in S_1 but not at $\boldsymbol{E}$, then $W(\boldsymbol{f}) < W(\boldsymbol{E})$;

(ii) S_1 contains no equilibrium point other than $\boldsymbol{E}$.

Further, since $W(\boldsymbol{E}) > 0$ the function $\boldsymbol{\phi}(\boldsymbol{f})$ is continuous at $\boldsymbol{E}$: hence we can now choose ρ_2 to be such that

(iii) if $\boldsymbol{f}$ is in S_2 then $\boldsymbol{\phi}(\boldsymbol{f})$ is in S_1.

Now $S_1 - S_2$ is a bounded closed region and does not contain $\boldsymbol{E}$: hence the greatest value m taken by $W(\boldsymbol{f})$ in $S_1 - S_2$ must be such that $m < W(\boldsymbol{E})$.

Suppose now that $\boldsymbol{f}(i)$ is in S_2 and near enough to $\boldsymbol{E}$ to make $W(\boldsymbol{f}(i)) > m$. Then, by (11), $W(\boldsymbol{f}(r)) > m$ for $r = i+1, i+2, \cdots$, and so $\boldsymbol{f}(r)$ can never lie in $S_1 - S_2$ if $r \geqq i$. However, $\boldsymbol{f}(i)$ is in S_1 and, by (iii), as r runs through $i+1$, $i+2, \cdots$ the first point $\boldsymbol{f}(r)$ (if any) to fall outside S_1 would be preceded by a point in $S_1 - S_2$. But this cannot occur: hence $\boldsymbol{f}(r)$ cannot escape from S_1. Since S_1 is bounded and closed, every limit point of $\{\boldsymbol{f}(r)\}$ is thus in S_1. By Part I and condition (ii) it follows that $\boldsymbol{E}$ is the sole limit point. Altogether, if $\boldsymbol{f}(i)$ is near enough to $\boldsymbol{E}$, then $\boldsymbol{f}(r) \to \boldsymbol{E}$ as $r \to \infty$, as required.

For Part III let us suppose that $\boldsymbol{E}$ is a limit point of $\{\boldsymbol{f}(r)\}$: we can then show that $\boldsymbol{E}$ is the unique limit of $\boldsymbol{f}(r)$ by modifying the argument used for Part II. We have to drop the requirement (i), and consequently cannot claim that $\boldsymbol{f}(r)$ is never in $S_1 - S_2$ for $r \geqq i$. However, we observe instead that it can be in $S_1 - S_2$ at most finitely often, since $\{\boldsymbol{f}(r)\}$ has no limit point in $S_1 - S_2$. Since $\boldsymbol{E}$ is a limit point of $\{\boldsymbol{f}(r)\}$ there are infinitely many points $\boldsymbol{f}(i)$ in S_2 and (thus also in S_1), and, as before, for every departure of $\boldsymbol{f}(r)$ from S_1 there is a point of the sequence in $S_1 - S_2$. Hence there can only be finitely many departures, and $\boldsymbol{f}(r)$ must be confined to S_1 when r is large enough. It follows as before that $\boldsymbol{f}(r) \to \boldsymbol{E}$ as $r \to \infty$.

6. We remark that better results than those in Theorem 3 can be obtained under various restrictions. Firstly, when there is a nondegenerate equilibrium $\boldsymbol{E}$ at which $W(\boldsymbol{f})$ has a strict maximum it is not difficult to show that this is the

only local maximum point: for any section of the "W-surface" by a plane passing through $\boldsymbol{E}$ will be parabolic with the vertex of the parabola at $\boldsymbol{E}$. Thus $\boldsymbol{E}$ must be the only stable equilibrium. In other cases there can be more than one stable equilibrium, necessarily degenerate: a simple case is

$$W(f_1, f_2, f_3) = [(f_1 - 2f_2)^2 + 3(f_1 + f_2 + f_3)^2]/10 = [3 + (f_1 - 2f_2)^2]/10,$$

with local maxima at (1, 0, 0) and (0, 1, 0).

Secondly, by Part III of Theorem 3, $\{\boldsymbol{f}(r)\}$ *must converge to a unique limit in all cases in which there are only isolated equilibria* (*or, equivalently, in which there is only a finite number of equilibria*). From the corollary to Theorem 1 and from the conditions for an equilibrium point [see (9) and (10) above] it is clear that every equilibrium point will be isolated unless one or more of certain determinants, formed from the numbers $w_{\alpha\beta}$, happen to vanish. In this sense any cases in which $\{\boldsymbol{f}(r)\}$ might not converge are exceptional, and we should be unlucky if we encountered such a case in a real situation.

Further, although in our deterministic model convergence to a "saddle point" equilibrium $\boldsymbol{E}$ is possible, yet such an equilibrium is unstable, and in practice random fluctuations would almost certainly lead the system to some point $\boldsymbol{f}'$, say, with $W(\boldsymbol{f}')$ appreciably greater than $W(\boldsymbol{E})$, and after $\boldsymbol{f}'$, since $W(\boldsymbol{f})$ must increase, $\boldsymbol{f}$ would be kept away from the neighborhood of $\boldsymbol{E}$. In view of all this Sewall Wright's picture of the process [see Section 5] is "practically" correct.

7. It only remains to prove the inequality (1). If we write $M_n = \boldsymbol{v}^T\boldsymbol{u}^n\boldsymbol{v}$, and exclude the trivial case $\boldsymbol{v}=0$, the inequality can be written

$$\psi(\boldsymbol{v}) = M_n/M_0 - (M_1/M_0)^n > 0 \tag{12}$$

under the conditions (α) all $v_i \geqq 0$, (β) all $u_{ij} = u_{ji} \geqq 0$, (γ) $\boldsymbol{v}$ is not a latent vector of $\boldsymbol{u}$. We know that there is a real orthogonal matrix, $\boldsymbol{R}$, say, which reduces $\boldsymbol{u}$ to principal axes, *i.e.*, $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$, $\boldsymbol{R}^T\boldsymbol{u}\boldsymbol{R} = \boldsymbol{t}$ where $t_{ij} = 0$ when $i \neq j$. Let $\boldsymbol{a} = \boldsymbol{R}^T\boldsymbol{v}$; then

$$M_n = \boldsymbol{v}^T\boldsymbol{u}^n\boldsymbol{v} = \boldsymbol{a}^T\boldsymbol{t}^n\boldsymbol{a} = \sum t_{\rho\rho}^n a_\rho^2, \tag{13}$$

where by condition (γ) $\boldsymbol{a}$ is not a latent vector of $\boldsymbol{t}$, *i.e.*, there exist i, j, such that

$$a_i \neq 0, \quad a_j \neq 0, \quad t_{ii} \neq t_{jj}. \tag{14}$$

Now M_n is the nth moment of a distribution of masses a_i^2 at respective distances t_{ii} from the origin. When all $t_{ii} \geqq 0$, (12) is a standard result, true even if (α) and/or (β) do not hold. For on allowing n to take all real values, a straightforward differentiation and use of Cauchy's inequality show that $(d/dn)^2 \ln M_n > 0$, *i.e.* the graph of $\ln M_n$ as a function of n is concave upwards. Hence of two adjacent chords of the graph the one on the right must have the greater slope; if $h < m < n$,

$$\frac{\ln M_n - \ln M_m}{n-m} > \frac{\ln M_m - \ln M_h}{m-h},$$

or

$$M_n^{m-h} M_h^{n-m} > M_m^{n-h}, \tag{15}$$

and (12) is the particular case $m=1$, $h=0$. (*A fortiori* (15) is true if n and h are even, even if not all $t_{ii} \geqq 0$.) However, if some t_{ii} are negative and n is odd (15) need not necessarily hold, even if (α) and (β) are satisfied. For example, take $v=\begin{bmatrix}2\\1\end{bmatrix}$, $u=\begin{bmatrix}0&1\\1&0\end{bmatrix}$; then $M_0=M_2=5$, $M_1=M_3=4$, and $M_3M_1<M_2^2$ and $M_3^2M_0<M_2^3$.

We now establish the truth of (12) under conditions (α), (β) and (γ) by *reductio ad absurdum*. Suppose it untrue; choose $\boldsymbol{u}$, $\boldsymbol{v}$ of the smallest possible dimensions k to satisfy (α), (β) and (γ) but not (12). Clearly $k>1$; and we can suppose all $v_i>0$, for if any $v_i=0$, then on suppressing the ith row and column of $\boldsymbol{u}$ we have *a fortiori* a contradiction of (12) in a smaller number of dimensions. Thus $\psi(\boldsymbol{v})$ depends only on the direction of the vector $\boldsymbol{v}$, is a continuous function of $\boldsymbol{v}$, is strictly positive on the boundary of the set of $\boldsymbol{v}$'s that satisfy (α), where some $v_i=0$, and is negative or zero at some point $\boldsymbol{v}^*$ that lies inside this set and also satisfies (γ). Thus among points satisfying (α) there is at least one point $\boldsymbol{V}$ at which the function $\psi(\boldsymbol{v})$ takes its minimum value $m \leqq 0$ and is stationary. If $m<0$ such a point $\boldsymbol{V}$ must satisfy (γ), since it is easy to see that $\psi(\boldsymbol{v})=0$ if $\boldsymbol{v}$ is a latent vector of $\boldsymbol{u}$. If $m=0$ we may clearly take $\boldsymbol{V}$ to be $\boldsymbol{v}^*$, and again $\boldsymbol{V}$ will satisfy (γ). Write $\psi(\boldsymbol{v})=\omega(\boldsymbol{a})$, where $\boldsymbol{a}=\boldsymbol{R}^T\boldsymbol{v}$, and write $\boldsymbol{A}=\boldsymbol{R}^T\boldsymbol{V}$: then at $\boldsymbol{a}=\boldsymbol{A}$ the condition (14) is satisfied and $\omega(\boldsymbol{a})$ is stationary, so that $\partial\omega(\boldsymbol{a})/\partial a_i=0$ at that point. But, by (13), $\partial M_n/\partial a_i = 2a_i t_{ii}^n$, so that for each i either $A_i=0$ or $M_0 t_{ii}^n - M_n - n(M_1/M_0)^{n-1}(M_0 t_{ii} - M_1)=0$ at $\boldsymbol{a}=\boldsymbol{A}$. In the latter case t_{ii} is a root of the equation

$$t^n - nt(M_1/M_0)^{n-1} + (n-1)(M_1/M_0)^n - \psi(\boldsymbol{V}) = 0. \tag{16}$$

The left-hand side of (16), considered as a function of t only, has a unique minimum for nonnegative t when $t=M_1/M_0$ and its value is then $-\psi(\boldsymbol{V}) \geqq 0$: thus the only possible nonnegative root of (16) is $t=M_1/M_0$, and so $t_{ii} \leqq M_1/M_0$ whenever $A_i \neq 0$. But $\boldsymbol{A}^T t \boldsymbol{A}/\boldsymbol{A}^T\boldsymbol{A} = M_1/M_0$, and so $\sum A_\rho^2 t_{\rho\rho} = (M_1/M_0)\sum A_\rho^2$, which can accordingly only be satisfied if $t_{ii}=M_1/M_0$ whenever $A_i \neq 0$. This contradicts (14) for $\boldsymbol{a}=\boldsymbol{A}$ and so the theorem is established.

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SEPARABILITY CONDITIONS FOR SOME SELF-ADJOINT PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction. Let L_{xy} be a linear partial differential operator with independent variables x and y. Denote by T a real transformation $x=x(u, v)$, $y=y(u, v)$ with finite, nonvanishing Jacobian. Let the transformed operator be given by $TL_{xy}=L_{uv}$. If L_{uv} can be written in the form

$$F(u, v)(L_u + L_v), \tag{1}$$

where L_u and L_v involve u alone and v alone, respectively, the operator L_{xy} and the equation $L_{xy}[\psi]=0$, are said to be separable in the coordinates u, v. We shall call u, v a separable system of coordinates if L_{xy} is separable in u, v.

Separability conditions and separable systems of coordinates for Laplace's equation, the Helmholtz equation and the Schrödinger equation are well known. (See, for example, [1] to [4].) While these constitute the most important cases from the point of view of applications, it is of interest that results can be obtained for more general types using only elementary methods. In this paper we shall discuss separability of the equations

$$L[\psi] = 0, \tag{2a}$$

$$L[\psi] + k\psi = 0, \tag{2b}$$

where

$$L[\psi] = (A\psi_x)_x + (B\psi_y)_x + (B\psi_x)_y + (C\psi_y)_y. \tag{3}$$

Here A, B and C are functions of x and y, k is a constant, and subscripts refer to differentiation. We shall determine necessary and sufficient conditions that (2a) and (2b) be separable in coordinates u, v and for the special case $A=C$, $B=0$ we shall determine the separable systems of coordinates.

We assume throughout that all functions are twice differentiable in a region R of the xy-plane and the corresponding region S of the uv-plane where S is defined by the transformation $T: x=x(u, v)$, $y=y(u, v)$. We further assume that (2a) and (2b) are elliptic, *i.e.*, $AC-B^2>0$ in R. The equations remain elliptic under the transformation T but not necessarily self-adjoint. However it is easy to show that the transformed equation becomes self-adjoint when multiplied by the Jacobian $J=\partial(x, y)/\partial(u, v)$.

2. Separability of (2a). We define

$$\begin{aligned} N[\psi] &= A\psi_x^2 + 2B\psi_x\psi_y + C\psi_y^2, \\ M[\psi, \phi] &= A\psi_x\phi_x + B(\psi_x\phi_y + \psi_y\phi_x) + C\psi_y\phi_y. \end{aligned} \tag{4}$$

In order that (2a) be separable in u, v the transformation T must be such that

$$\begin{aligned} N[u] &= F(x, y)U_1, \qquad M[u, v] = 0, \qquad N[v] = F(x, y)V_1, \\ L[u] &= F(x, y)U_2, \qquad L[v] = F(x, y)V_2, \end{aligned} \tag{5}$$

where F is an arbitrary function of x, y and where here and in what follows, U_i are functions of u alone, V_i are functions of v alone, $i=1, 2, 3, 4$. We assume that U_i, V_i are different from zero for all i.

The transformation T when applied to (2a) gives, after multiplying by J,

$$(A'\psi_u)_u + (B'\psi_v)_u + (B'\psi_u)_v + (C'\psi_v)_v = 0, \tag{6}$$

where $A' = JN[u]$, $B' = JM[u, v]$, $C' = JN[v]$. Hence conditions for separability are also

$$JN[u] = U_3V_3, \qquad M[u, v] = 0, \qquad JN[v] = U_4V_4. \tag{7}$$

This latter set of conditions is equivalent to (5) in the sense that (7) implies (5). This equivalence can be established without difficulty by making use of the self-adjoint property of (6), namely

$$\frac{\partial}{\partial u}(JN[u]) = JL[u], \qquad \frac{\partial}{\partial v}(JN[v]) = JL[v].$$

Let $\bar{u}$, $\bar{v}$ be defined as follows: $\bar{u} = \int U_1^{-1/2}du$, $\bar{v} = \int V_1^{-1/2}dv$. We now prove the following theorem:

THEOREM 1. *The necessary and sufficient conditions that* (2a) *be separable in* u, v *are* 1) $\bar{u}$ *and* $\bar{v}$ *satisfy the Beltrami equations*

$$A\bar{u}_x + B\bar{u}_y = W\bar{v}_y, \qquad B\bar{u}_x + C\bar{u}_y = -W\bar{v}_x \tag{8}$$

with $W = (AC - B^2)^{1/2}$, *and* 2) *functions* $U(u)$, $V(v)$ *exist such that*

$$W = U(u)V(v). \tag{9}$$

We first prove the conditions are necessary. From (5) we have $N[\bar{u}] = N[\bar{v}] = F$, $M[\bar{u}, \bar{v}] = 0$, and from these the Beltrami equations follow at once. Then from the Beltrami equations we get

$$N[\bar{u}] = W(\bar{u}_x\bar{v}_y - \bar{u}_y\bar{v}_x) = HW(U_1V_1)^{-1/2}$$

with $H = 1/J$. Combining this result with equations (7) gives $JN[\bar{u}] = U_3V_3U_1^{-1} = W(U_1V_1)^{-1/2}$ and similarly $JN[\bar{v}] = U_4V_4V_1^{-1} = W(U_1V_1)^{-1/2}$. Hence, (9) follows immediately.

To show that the conditions are sufficient, we note first that the equations $N[\bar{u}] = N[\bar{v}] = F$, $M[\bar{u}, \bar{v}] = 0$, follow from the Beltrami equations. Moreover, as above, we have $N[\bar{u}] = HW(U_1V_1)^{-1/2} = HU(u)V(v)(U_1V_1)^{-1/2}$ and a similar equation for $N[\bar{v}]$. These equations are obviously equivalent to (7) which completes the proof of the theorem.

3. Separability of (2b). If we transform (2b) by means of T, multiply by J and make use of (6) and (7), we get

$$\frac{1}{U_4}(U_3\psi_u)_u + \frac{1}{V_3}(V_4\psi_v)_v + \frac{k\psi}{N[u]} = 0. \tag{10}$$

Thus we have the theorem:

THEOREM 2. *The necessary and sufficient conditions that* (2b) *be separable in* u, v *are that* $\bar{u}$, $\bar{v}$ *satisfy* (8), U, V *exist which satisfy* (9) *and* $N[\bar{u}]$ *is such that*

$$1/N[\bar{u}] = f(u) + g(u). \tag{11}$$

We omit the proof which follows readily from Theorem 1 and proceed to discuss the special case $A=C$, $B=0$. This is a special case that occurs frequently in physical applications, *e.g.*, in the study of vibrations of nonhomogeneous membranes or flow of heat in substances whose thermal conductivity is a function of position.

4. Differential equations for f, g, $F=\log U$, $G=\log V$. With $A=C$, $B=0$ the separability conditions of Theorems 1 and 2 become

$$u_x = v_y, \quad u_y = -v_x, \quad A(x, y) = UV, \quad J = UV(f(u) + g(v)), \tag{12}$$

where we have omitted the bars from u and v. Thus we shall be concerned with the transformations $T: x=x(\bar{u}, \bar{v})$, $y=y(\bar{u}, \bar{v})$, and this involves no loss in generality.

Let $z=x+iy$ be an analytic function of $w=u+iv$, and write $z=\phi(w)$. Two transformations T_1, T_2 are then defined by $z_1=\phi_1(w)$, $z_2=\phi_2(w)$. We shall say that two such transformations are of the same type if ϕ_1 and ϕ_2 are related in the following manner:

1) $z_1=az_2+b$, $a\neq 0$,
2) $w_1=\alpha w_2+\beta$, $\alpha\neq 0$, α real or purely imaginary.

We have $J=x_u^2+x_v^2=|\phi'(w)|^2$ and

$$2\log|\phi'(w)| = F(u) + G(v) + \log(f(u) + g(v))$$

with $F(u)=\log U(u)$, $G(v)=\log V(v)$. Since $\log|\phi'(w)|$ is a harmonic function we find that

$$(F'' + G'')(f + g)^2 + (f + g)(f'' + g'') - (f'^2 + g'^2) = 0, \tag{13}$$

where here and in what follows, primes indicate differentiation. We omit the details, which amount to straightforward differentiation and integration, and find from (13) that

$$F'' = m^2f^2 - \frac{nf}{2} + n_1, \qquad G'' = -m^2g^2 - \frac{ng}{2} - n_1, \tag{14}$$

$$\begin{aligned} f'^2 + m^2f^4 - nf^3 + p^2f^2 - qf + r &= 0, \\ g'^2 - m^2g^4 - ng^3 - p^2g^2 - qg - r &= 0, \end{aligned} \tag{15}$$

where m, n, n_1, p, q, r are real constants. These equations determine, to within arbitrary constants, the F, G, f and g. We note that putting $f=-g$, $du=idv$, changes the first of (15) into the second and vice versa. Hence we need consider only one of (15) and we choose the one involving g.

It is convenient to define $h(g)=m^2g^4+ng^3+p^2g^2+qg+r$ and write the second of (15) as

$$g'^2 - h(g) = 0. \tag{16}$$

5. Classification of solutions of (14) and (15). We shall discuss only those solutions which are real in both f and g. Constants of integration will be designated by $a_i, b_i, c_i, u_i, v_i, f_i, g_i$ with $i=1, 2, \cdots$, and are to be real. The function $n_1(u^2-v^2)/2+a_1u+b_1v+c_1$ will be denoted by $R(u, v)$. We assume $n_1\neq 0$; the extension to $n_1=0$ is obvious. We shall not discuss all possible cases that arise from the solutions of (14) and (15) but only those which seem of most interest and simply state the results for the others.

Case I. $m=n=0$, $\mu^2=-n_1/4$. Define complex constants γ and δ by $\gamma=(a_1-ib_1)/2$, $\delta=$ any complex constant whose real part is $c_1/2$. From (14) we find

$$F(u) = -2\mu^2u^2 + a_1u + a_2, \quad G(v) = 2\mu^2v^2 + b_1v + b_2, \; a_2 + b_2 = c_1.$$

(a) $p=q=r=0$. Here $2 \log |\phi'(w)| = R(u, v)+\log (f_1+g_1)$ and

$$\phi(w) = (f_1 + g_1) \int^w \exp (-\mu^2\zeta^2 + \gamma\zeta + \delta)d\zeta.$$

Hence we say that all transformations here are of the type

$$\phi(w) = \int^w \exp (-\zeta^2)d\zeta.$$

(b) $p=0$, $q\neq 0$. In this case we find

$$\phi(w) = C \int^w (\zeta + \zeta_1) \exp (-\mu^2\zeta^2 + \gamma\zeta + \delta)d\zeta,$$

where C and ζ_1 are complex constants. The transformations are of the type

$$\phi(w) = \int^w (\zeta + \zeta_2) \exp (-\zeta^2)s\zeta.$$

(c) $p\neq 0$, $h=p^2(g-g_1)^2$. Transformations are of type (a).

(d) $p\neq 0$, $h=p^2(g-g_1)(g-g_2)$, g_1 and g_2 real and distinct. All transformations here can be shown to be of the type

$$\phi(w) = \int_0^w \sin (\sigma_1\zeta + \zeta_1) \exp (-\zeta^2)d\zeta,$$

where σ_1 and ζ_1 are complex constants.

It is of interest to notice that the cases where F and G are both constant,

which implies $n_1=0$, are the cases in which the separable coordinate systems are the four cylindrical coordinate systems found by Eisenhart [2]. These are the only separable systems for $A(x, y)=\text{const.}$

Case II. $m=0$, $n\neq 0$.

(a) $h=n(g-g_1)^3$. Transformations are of type I(b).

(b) $h=n(g-g_1)^2(g-g_2)$. Transformations are of type I(d).

(c) $h=n(g-g_1)(g-g_2)(g-g_3)$, g_1, g_2, g_3 real and $g_1>g_2>g_3$. The solution here can be found in terms of elliptic functions.* We have

$$g=\{g_1-g_2\,\mathrm{sn}^2[M(v+v_1), k_1]\}\,\mathrm{cn}^{-2}[M(v+v_1), k_1],$$

where $k_1^2=(g_2-g_3)/(g_1-g_3)$ and $M=\frac{1}{2}n(g_1-g_3)$, and

$$f=(g_1-g_2)\,\mathrm{sn}^2[M(u+u_1), k_2]-g_1$$

with $k_2^2=1-k_1^2$. In the remainder the modulus k_1 is to be associated with v and k_2 with u. Explicit reference to these moduli will be omitted. Also for brevity we write v for $v+v_1$ and u for $u+u_1$. Thus we have

$$f+g=(g_1-g_2)\,|\,\mathrm{sn}\,Mw\,|^2(1-\mathrm{sn}^2 Mv\ \ \mathrm{dn}^2 Mu)/\mathrm{cn}^2 Mv,$$

where $w=u+iv$ and the modulus of sn Mw is k_2.

The functions F and G can be expressed in terms of the theta functions, thus:

$$F=\mu^2u^2+a_1u+a_2+2\log\theta(Mu, k_2),$$
$$G=-\mu^2v^2+b_1v+b_2+2\log\theta(iMv, k_2),$$

where $\mu^2=n_1/2+(n/4)[g_3+(g_1-g_3)E/K]$, and K and E are complete elliptic integrals.

Finally we find that

$$2\log|\phi'(w)| = \mu^2(u^2-v^2)+a_1u+b_1v+c_1 + \log[(g_1-g_2)\cdot|\,\mathrm{sn}\,(Mw)\cdot\Theta(Mw)\,|^2\Theta(0)^2],$$

and hence the transformations for this case are of the type

$$\phi(w)=\int^w\Theta(\sigma_1\zeta+\zeta_1)\,\mathrm{sn}\,(\sigma_1\zeta+\zeta_1)\exp(-\zeta^2)d\zeta.$$

(d) $h=n(g-g_1)[(g-a)^2+b^2]$, a and b real constants. This case can also be expressed in terms of the elliptic functions. We omit the details and give only the result that the transformations are of the type

$$\phi(w)=\int^w\{\Theta(\sigma_1\zeta+\zeta_1)[1-\mathrm{cn}\,(\sigma_1\zeta+\zeta_1)]\}^{1/2}\exp(-\zeta^2)d\zeta.$$

* The necessary details concerning elliptic functions can be found in [5].

Case III. $m \neq 0$.

(a) $h = m^2(g-g_1)^4$. Transformations are of type I(a).

(b) $h = m^2(g-g_1)^3(g-g_2)$. Transformations are of type I(b).

(c) $h = m^2(g-g_1)^2(g-g_2)^2$. Transformations are of type I(a).

(d) $h = m^2(g-g_1)^2[(g-a)^2+b^2]$. Transformations are of type I(a).

(e) $h = m^2(g-g_1)(g-g_2)(g-g_3)^2$. Transformations are of type I(d).

(f) $h = m^2(g-g_1)(g-g_2)(g-g_3)(g-g_4)$. This case can be shown to yield transformations that are of type II(c).

(g) $h = m^2(g-g_1)(g-g_2)[(g-a)^2+b^2]$. This case can be shown to yield transformations that are of type II(d).

The results of this section can be summarized as follows: The transformations which separate (2b) with $A = C$, $B = 0$ and which are defined by the real and imaginary parts of $z = \phi(w)$ with $z = x+iy$ and $w = u+iv$, are of the type

$$\phi(w) = \int^{w} \chi(\zeta) \exp(-\zeta^2) d\zeta, \tag{17}$$

where $\chi(\zeta)$ assumes one of the values, 1) 1, 2) $\zeta+\zeta_1$, 3) $\sin(\sigma_1\zeta+\zeta_1)$, 4) $\Theta(\sigma_1\zeta+\zeta_1)$ $\mathrm{sn}(\sigma_1\zeta+\zeta_1)$, 5) $\{\Theta(\sigma_1\zeta+\zeta_1)[1+\mathrm{cn}(\sigma_1\zeta+\zeta_1)]\}^{1/2}$.

The coefficient $A = A(x, y) = U(u(x, y), v(x, y))$ are those for which separable transformations exist. Although all transformations which are separable are given by (17) it is obvious that an explicit expression for A cannot be written in most cases.

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AN OPTIMAL SEARCH PROCEDURE*

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The problem which we are concerned with is the following. We are given a set of objects, one of which has been previously selected without our knowledge, as the one which we must determine. In addition, we have assigned to each object a probability of having been selected, the sum of all such probabilities being unity. We must determine the selected object by a series of yes-no questions, arbitrarily dividing the set of objects into two groups and asking which it is contained in. We continue this yes-no process until the selected object is known. For example, we might use the yes-no process to determine which letter of the alphabet was randomly selected from a book, given the probabilities (the frequencies) of each letter. We would now like to know what the best strategy of division is, where by a strategy we mean a plan which indicates exactly what the next division will be, given that the selected object is contained in any particular group of the original objects. For example, given four objects with assigned probabilities (.4), (.3), (.2), (.1), our strategy might be, first divide in the following way: (.4, .1), (.3, .2). Then, if the object is learned to be in the left-hand group, further divide, (.4), (.1), and if it is in the right-hand group, divide, (.3), (.2). The expected value, that is the probable number of divisions for this strategy, would be two.

The questions which will now be answered are: Given any set of objects, what strategy should be used to obtain the smallest expected value (the optimal strategy)? What is the smallest expected value?

It is important to observe that although this seems like a method of repeated division, it can be viewed rather as one of combination. We could start with the entire set as in the above strategy, and describe where the first division, and subsequent divisions dependent upon the outcome of the preceding ones, are going to be made. Instead, we can proceed in precisely the reverse manner, beginning with the several isolated objects and indicating a series of repeated combinations, the first few of which correspond to the last few divisions, and the final one to the first division. Since in our strategy divisions are always into two groups, each of the combinations will similarly involve two groups.

In the above example then, the method of combination corresponding to the strategy used would be one of the following two. First, combine the two objects with the probabilities (.4) and (.1); next combine those with probabilities (.3) and (.2). Finally combine the two groups corresponding to (.4, .1) and (.3, .2). The second method of combination would simply contain a reversal of the first two steps of the preceding one. It is seen from this example that a particular strategy need not have a unique method of combination. However, it should be fairly evident that any method of combination will always have only one division strategy corresponding to it. Therefore, in order to determine

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the optimal strategy of division of a set of objects, it is only necessary to determine a corresponding method of combination.

A concept of importance is that of the n-set. If we are given a set of n objects, then we call the array of the corresponding probabilities, arranged from left to right in descending order of magnitude, an n-set. This left to right arrangement has no significance except that it will be convenient later on. When we combine two of the objects of the n-set, and assign the sum of the two probabilities as the probability of the new "object", we have essentially reduced the n-set to an $(n-1)$-set. Any method of combination is then a way of making a combination of two objects $n-1$ times until the 1-set is reached.

Another concept which we will use is that of the partial sum. At any stage of combination, the sum of probabilities of all the pairs of combined objects we call the partial sum. The probability of one of the original objects can be included more than once in a partial sum, since after it has been combined to form a new object, this latter object may again be combined.

We can now state the main theorem, although it will not be proved until later, following a series of lemmas.

THEOREM. *The optimal rule of combination is: At each stage combine the objects with the two smallest probabilities.*

LEMMA 1. *The method of combination of any n-set is, after the first step, precisely the same as that of some $(n-1)$-set.*

Proof. In the method of combination of the n-set, there must be a first combination. Therefore, the $(n-1)$-set equivalent to the n-set, except for the replacement of these two objects by the combined one, has, from then on, the same method of combination as the n-set. We have, in fact, referred to the n-set as being reduced to the $(n-1)$-set by the combination of the two objects. It follows that the expected value of the n-set should be that of the $(n-1)$-set plus the probability that we will have to ask another question if the object is contained in the two combined ones. Since the latter is exactly the sum of the two original probabilities, we can write the equations

$$E_r = E_{r-1} + (p_m + p_n), \qquad E_1 = 0,$$

where E_r is the expected value of a given r-set, and p_m and p_n are the probabilities of the combined objects. If we proceed by induction, the result is that E_n is the sum of $n-1$ combinations of probabilities and the problem is really how to combine probabilities two at a time so that the total of these probabilities is a minimum. Each original probability is going to be combined one or more times.

LEMMA 2. *In an optimal method of combination, a larger probability cannot be combined more times than a smaller one.*

Proof. If we had a method whereby a larger probability were combined more

often than a smaller one, we could interchange the two probabilities in our operations and get a smaller expected value.

LEMMA 3. *We are given the optimal value of an n-set, and any two probabilities p_r and p_s where $p_r > p_s$. If we increase p_r by any amount k, and decrease p_s by the same amount, the optimal value of the transformed set cannot be greater than that of the original.*

Proof. By Lemma 2, p_r cannot be combined a greater number of times than p_s and so if we use the same method of combination for the transformed set we can add plus k no more than minus k. It is impossible to get a greater expected value in this way. In addition, this may not even be the optimal method for the transformed set.

We are now in a position to give an inductive proof of the main theorem.

Proof of the main theorem. The rule obviously holds for all the 1-sets, of which there is only one. We shall now assume that it holds for every r-set, where r assumes the values $1, \cdots, n-1$. If it can be shown that the rule is true for any n-set, then the theorem is proved, by induction.

We are given an arbitrary n-set, arranged in descending order of size, $p_n, \cdots, p_1$. There are three possible first combinations:

(1) A combination of objects r and s, that is, an addition of p_r and p_s, where neither r nor s is equal to 1 or 2.

(2) A combination of r and s where one of r and s is equal to either 1 or 2.

(3) A combination of 1 and 2.

It must be shown that the third method is at least as good as the other two.

(1) Combining r and s gives a partial sum of (p_r+p_s) and the $(n-1)$-set $p_n, \cdots, (p_r+p_s), \cdots, p_1$. The next combination must be 1 and 2 according to the inductive assumption, giving a partial sum of $(p_r+p_s+p_1+p_2)$ and the $(n-2)$-set $p_n, \cdots, (p_r+p_s), \cdots, (p_1+p_2), \cdots, p_3$.

If, instead we first combine 1 and 2, then we can next combine r and s, which may not even be optimal. We arrive at the same partial sum and the same $(n-2)$-set. This method is at least as good as combining r and s in the n-set.

(2a) If 1 and r $(r>3)$ are combined, our partial sum is (p_1+p_r) and we have the $(n-1)$-set $p_n, \cdots, (p_1+p_r), \cdots, p_2$. The next combination must be (p_2+p_3); this gives a partial sum of $(p_1+p_2+p_3+p_r)$ and an $(n-2)$-set $p_n, \cdots, (p_1+p_r), \cdots, (p_2+p_3), \cdots, p_i$, where $p_i=p_4$ if $r\neq 4$.

If, instead, 1 and 2 are combined, we have a partial sum of (p_1+p_2) and an $(n-1)$-set $p_n, \cdots, (p_1+p_2), \cdots, p_3$. The next combination can be 3 and r, which may not even be optimal. This gives the same total partial sum as before and an $(n-2)$-set, $p_n, \cdots, (p_3+p_r), \cdots, (p_1+p_2), \cdots, p_i$. By Lemma 3, the

new $(n-2)$-set has an expected value no greater than the original, and since the partial sums are equal, this method must be at least as good as the former combination.

(2b) If 1 and r are combined, where r is 3, then one of two situations can arise. If $(p_1+p_2)>p_4$ the proof is analogous to (2a). If however, $(p_1+p_2)<p_4$, then, after we have combined 1 and 3 we would combine 2 and the combination of 1 and 3. In the second method, after we had combined 1 and 2, we would combine 3 with the combination of 1 and 2. The resulting $(n-2)$-sets would be the same, and the partial sum of the 1 and 2 combination would be lower than that of the 1 and 3 combination. Therefore the latter would be at least as good as the former method.

In the proof of (2) it is possible to interchange the values for p_1 and p_2 so as to prove the theorem for the combination of 2 and r. All possible first combinations in the n-set have been treated and hence the theorem is proved.

A rather interesting situation is that of nonuniqueness. If, at any stage of the method, there exist two or more probabilities which are equal, we can arrange them in any order we like, as long as they are adjacent. Further, if these probabilities represent not individual objects but groups, then the order will affect the final combination procedure. The result will be two or more arrangements which are equally acceptable.

As final remarks, let me indicate the outcome of the example presented at the outset, that of the determination of a letter of the alphabet. We are given a set of probabilities corresponding to each letter and the blank space. They range in size from .001 for the letter Q to .200 for the blank. Using the rule of combination, the expected number of divisions or questions is 4.154.

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SOLUTION OF A SET OF GAMES

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1. Introduction. This paper deals with a family of zero-sum two-person games in the Von Neumann sense [1]. The rules of the games are simple, but their implications are sufficiently obscure that the methods of game-theoretic analysis are necessary to determine the solutions. In this respect, the games fall into a category which, at present, has too few representatives known; namely, those games which (unlike tic-tac-toe) are not so simple that a theoretical analysis is superfluous, and (unlike chess) not so complicated that the analysis is inadequate to provide the solution.

2. Description of the games. The games considered are known by the designation of the "n-coin game," where n is a positive integer. The particular case of the three-coin game will be used to illustrate the rules of play; the general case is completely analogous.

In the three-coin game, each player A and B is supplied initially with three identical coins or other counters.

First Move. Each player chooses, unknown to the other, an integer between 0 to 3 (inclusive) and places that number of his coins to his right hand, which is thereafter kept closed. The number of coins selected by A and B, respectively, in this move will be denoted A_1 and B_1.

Second Move. Player A attempts to guess the total number of coins now held in both player's right hands. He announces his guess, A_2, an integer between 0 and 6.

Last Move. Player B, having heard A's second move, now attempts himself to guess the total. His guess, B_2, must dfffer from A_2.

The hands are now opened and the correct total ascertained. If either player has guessed correctly, he wins one arbitrary unit from the other. Otherwise, the play is a draw. In practice, when a draw occurs, it is customary to repeat the game with A and B exchanging roles, until one or the other wins. However, this convention need not affect our theoretical studies, which are concerned simply with optimal strategies for A and B in a single play of the game.

3. Heuristic discussion. Each player has some benefits under the rules, and it is not immediately obvious which has the greater. Player A has the choice of seven numbers (0, 1, 2, 3, 4, 5, or 6) for his second move, while B has only six. Moreover, if A guesses the sum correctly, then B's last move is completely futile, since he is defeated already. However, B has the advantage of hearing A_2 before naming B_2, and A_2 may give B some clue as to A_1, A's choice at the first move. For example, if $A_2=6$, then B can be reasonably sure that A's right hand holds three coins. This information will be useless in the case when B is also holding 3, but in any other case, B can be assured of a win.

Note that A can cancel this advantage to B by making a guess of $A_2=3$ at his second move. This reveals nothing to B about A's first move. On the other hand, in order to utilize this method of annulling his opponent's advantage, A must sacrifice his own advantage of complete freedom of choice at Move 2.

The relative importance of each of these features of the game will be revealed in the theoretical analysis below.

4. A simple special case. Before undertaking the analysis of the general n-coin game, we shall consider some special cases. Our first example is the one-coin game. Observe first, that among the possible strategies for either player, there are some which offer no possibility of winning. For example, for A to play $A_1=1$, $A_2=0$ is manifestly absurd. It is possible to exclude such strategies by employment of the dominance principle [1], but it is hardly necessary to use

so sophisticated a technique to effect so elementary a result. In the remainder of this paper, only strategies which give the player a possibility of success ("feasible" strategies), will be considered. This does not destroy the generality of the results, since, as remarked above, all others may be excluded by dominance. Henceforth, we shall use the term strategy to denote feasible strategy, unless otherwise specified. With this convention, A has exactly four strategies:

A–I : $A_1 = 0,\ A_2 = 0,$ A–III: $A_1 = 1,\ A_2 = 1,$

A–II : $A_1 = 0,\ A_2 = 1,$ A–IV: $A_1 = 1,\ A_2 = 2;$

and B has only two strategies

$$B\text{–I:}\quad B_1 = 0,\ B_2 = \begin{cases} x \\ 0 \\ 1 \end{cases} \text{ as } A_2 = \begin{cases} 0 \\ 1, \\ 2 \end{cases}$$

$$B\text{–II:}\quad B_1 = 1,\ B_2 = \begin{cases} 1 \\ 2 \\ x \end{cases} \text{ as } A_2 = \begin{cases} 0 \\ 1. \\ 2 \end{cases}$$

In case $B_1 = 0$ and $A_2 = 0$, B presumably has a lost game, and his second move is immaterial; a similar comment holds for the case $B_1 = 1$, $A_2 = 2$. These cases are denoted above by the (purely arbitrary) notation $B_1 = x$. As a matter of practical expediency, B may be best advised to take $B_2 = 1$ in these two cases since this will give him an opportunity to win in the unlikely event that A is playing one of the nonfeasible strategies.

The payoff matrix for the game is given in Table 1.

TABLE 1. PAYOFF FOR THE ONE-COIN GAME (PAYOFF TO A)

	B-I	B-II
A-I	1	−1
A-II	−1	1
A-III	1	−1
A-IV	−1	1

From this table, it can be seen that basic optimal strategy mixes [3] for A include (and are limited to)

I and II in equal proportions,
I and IV in equal proportions,
II and III in equal proportions,
III and IV in equal proportions;

and B has only one optimal strategy, which is to mix B-I and B-II in equal

proportions. The value of the game is zero; no play will end in a draw, and neither player can protect himself against loss if he attempts to take advantage of an opponent's error. In several of these respects, we shall see that the one-coin game is atypical.

5. The two-coin game. When $n=2$, a game is obtained which is typical in all important respects of all higher values of n, and still has few enough possibilities that detailed examination is practical.

It will be convenient to extend our notation as follows: Let $\overline{B}_1$ be player A's guess of B's first move, B_1, *i.e.*, $\overline{B}_1=A_2-A_1$. Then A's strategies in the two-coin game are nine in number and may be tabulated as in Table 2.

TABLE 2. STRATEGIES FOR PLAYER A IN THE TWO-COIN GAME

Strategy number	$\overline{B}_1$	A_1	A_2
A-I	0	0	0
A-II	0	1	1
A-III	0	2	2
A-IV	1	0	1
A-V	1	1	2
A-VI	1	2	3
A-VII	2	0	2
A-VIII	2	1	3
A-IX	2	2	4

The strategies for Player B may be described by the following convention. Any strategy for B determines a column in the payoff matrix, provided the A-strategies are listed down the side as in Table 2. Conversely, any column of nine entries of $+1$, -1, or 0 determines a B-strategy provided certain conditions are observed. The restrictions are stated below for the two-coin game. The form these take in the general case (with parameter n) is given in brackets.

Condition 1. Let the A-strategies be classed into the following three $[n+1]$ mutually exclusive sets:

All strategies for which either $\overline{B}_1=0$, or $\overline{B}_1=1, \cdots$, or $\overline{B}_1=n$. Then there must be $+1$ entries in the column defining the B-strategy against all A-strategies in exactly one of these sets, and no other $+1$ entries may appear. Since any B-strategy whatsoever must have a value of B_1, one of the three $[n+1]$ estimates $\overline{B}_1$ must be correct, *i.e.*, lead to a payoff to A of $+1$. We shall denote a B-strategy in which $B_1=i$ as a strategy of Type i.

Condition 2. The remaining 6 $[2(n+1)]$ entries in the column may be filled with 0 and -1 entries, but one and only one -1 entry* may be inserted against

* If nonfeasible B-strategies are considered, this condition should be modified to read " . . . not more than one -1 entry . . . ."

any single value of A_2. Thus, strategy A-VI and A-VIII both display $A_2=3$. In the column defining the B-strategy (say of Type 0), player B must select his call B_2 corresponding to $A_2=3$. He may select it so that he will win against A-VI, *i.e.*, $B_2=2$, or against A-VIII, *i.e.*, $B_2=1$, but he cannot choose a strategy which will win against both.

Brief consideration will establish that any column conforming to conditions (1) and (2) above will define a B-strategy, and conversely. It is not difficult to determine all B-strategies in the two-coin game, of which there are exactly ten. These strategies and the payoff matrix for the game appear in Table 3. (In this table $+1$ and -1 are abbreviated to $+$ and $-$ respectively, for brevity.) The reader is urged to construct Table 3 for himself. By so doing, he will greatly facilitate his understanding of the proofs given in subsequent sections.

TABLE 3. PAYOFF MATRIX, TWO-COIN GAME (PAYOFF TO A)

A-Strategies			B-Strategies									
Strategy number	$\bar{B}_1$	A_2	B-I	B-II	B-III	B-IV	B-V	B-VI	B-VII	B-VIII	B-IX	B-X
			Type 0				Type 1		Type 2			
A-I	0	0	+	+	+	+	−	−	−	−	−	−
A-II	0	1	+	+	+	+	−	−	0	−	−	0
A-III	0	2	+	+	+	+	−	0	−	−	0	0
A-IV	1	1	−	−	−	−	+	+	−	0	0	−
A-V	1	2	−	−	0	0	+	+	0	0	−	−
A-VI	1	3	−	0	0	−	+	+	−	−	−	−
A-VII	2	2	0	0	−	−	0	−	+	+	+	+
A-VIII	2	3	0	−	−	0	−	−	+	+	+	+
A-IX	2	4	−	−	−	−	−	−	+	+	+	+

A somewhat tedious but straightforward examination of this payoff matrix suffices to establish the following results.

PROPOSITION 1. *The game is fair.* [2]

PROPOSITION 2. *Player B has several optimal mixed strategies. Every one of the ten B-strategies is active in at least one of the optimal strategy mixes available to B.*

PROPOSITION 3. *Player A has only one optimal mixed strategy; namely, A-III, A-V, and A-VII in equal proportions.*

6. The general theorems. Statements analogous to Propositions 1, 2, and 3 hold in the general case. We shall present proofs of these general theorems. Because of the awkward nature of the operations, it will prove convenient to present the proofs in the language of a specific game, and we shall use $n=3$, the three-coin game, for this purpose. However, it will be observed that the methods we employ may immediately be applied to any value of n.

THEOREM 1. *If $n>1$, the n-coin game is fair.*

Proof for the three-coin game. Using notation like that introduced in the preceding section, we first show a mixed strategy for B in which the sum of the entries in any row is nonpositive. This implies that if the strategies so selected are played in equal proportions against any A-strategy, pure or mixed, the expected payoff to A will not exceed zero [2].

Select any nonnegative integers, $i\neq j$ $(i, j\leqq 3)$. We now choose B-strategies of Type i and j. In the column defining the Type i strategy, we insert -1 against every A-strategy which has $\overline{B}_1=j$; similarly, the Type j strategy has -1's inserted in those positions where $\overline{B}_1=i$. The other -1 entries are immaterial as long as Condition 2 is observed. It is obvious that such strategies satisfy the Conditions 1 and 2 of the preceding section and assure that the row sums are nonpositive. B can now play these two pure strategies with equal weight in a mixed strategy. This completes the proof that B can force an expected payoff of 0 or less.

Player A can force an expected payoff of 0 or more by playing an equal weight mix of the four stragegies:

$$\overline{B}_1 = 0,\ A_2 = 3; \qquad \overline{B}_1 = 1,\ A_2 = 3; \qquad \overline{B}_1 = 2,\ A_2 = 3; \qquad \overline{B}_1 = 3,\ A_2 = 3.$$

By Condition 1, every B-strategy must lose against exactly one of these four. Also, according to Condition 2, each B-strategy will win against one and only one of them. Therefore, any column sum is zero.* This completes the proof. The restriction $n>1$ is, of course, unnecessary since we have already proved the one-coin game to be fair. However, the method of proof used here requires $n>1$.

THEOREM 2. *In the n-coin game, any pure B-strategy is active in some mixed optimal strategy for B.*

Proof for the three-coin game. Let S be a given pure B-strategy. We shall produce a four-strategy mix of B-strategies including S which has all row sums nonpositive.

Strategy S is of some type, say i; rename $S=S_i$, and begin construction of the other three strategies in the mix by choosing them to be one of each of the three types among Types 0, 1, 2, and 3 obtained by omitting Type i. Name each strategy by a subscript defining its type, *i.e.*, S_0, S_1, S_2, and S_3, of which one is S and is completely defined, and the others still have -1 entries to be inserted in the columns of the payoff matrix defining them.

We shall insert these -1 entries such that each row has at least one such -1 entry. Since each row (defined by $\overline{B}_1$ and A_2) has exactly one $+1$ entry (by construction) the four strategies S_0, S_1, S_2, and S_3 may be played with equal weight, giving nonpositive payoff to A. This will prove the theorem. We will assign these -1 entries systematically. For each value of A_2, there are one or more A-

* If nonfeasible B-strategies are considered, any column sum is nonnegative.

strategies, *i.e.*, one or more rows in the matrix. Table 4 shows the number of occurrences of each A_2 value.

TABLE 4. OCCURRENCES OF VALUES OF A_2 IN FEASIBLE A-STRATEGIES IN THE THREE-COIN GAME

Values of A_2	0	1	2	3	4	5	6
Number of occurrences	1	2	3	4	3	2	1

We begin by assigning the four -1's which fall on rows where $A_2=3$. One of these already is entered, in the S_i column (since it is a feasible strategy). This lies on a row for which $\overline{B}_1=j$ (say), $j\neq i$. In column S_j insert a -1 against an A-strategy for which $A_2=3$, and $B_1\neq i, j$, say at $\overline{B}_1=k$. This is always possible since for each $\overline{B}_1$, an A-strategy exists with $A_2=3$. Next, in column S_k, insert -1 against an A-strategy for which $A_2=3$, $\overline{B}_1\neq i, j, k$, say at $\overline{B}_1=m$. Lastly, in column S_m, insert a -1 against strategy $A_2=3$, $\overline{B}_1=i$. It is obvious that this construction is always possible and complies with Conditions 1 and 2.

In similar manner, three -1 entries are made against A-strategies with $A_2=4$. One of these is already determined in column S_i. Suppose it occurs against the A-strategy with $\overline{B}_1=j'$. In column $S_{j'}$, insert a -1 against an A-strategy with $A_2=4$, $\overline{B}_1\neq i$, say at $B_1=k'$. This is always possible. In column $S_{k'}$, insert a -1 against an A-strategy with $A_2=4$. This may or may not occur where $\overline{B}_1=i$, but it, too, is always possible, since three distinct values of $\overline{B}_1$ ($\overline{B}_1=1$, 2, and 3) permit A-strategies with $A_2=4$.

Similarly, the three -1 entries for $A_2=2$, the two entries for $A_2=1$, and the two entries for $A_2=5$ can be assigned. The one entry for $A_2=0$ is inserted in any one (or more) of columns S_1, S_2, and S_3; the entry for $A_2=6$ is inserted in any one (or more) of columns S_0, S_1, and S_2. This completes the construction. It can readily be seen that the required conditions are satisfied and the theorem proved.

THEOREM 3. *If $n>1$, in the n-coin game, no optimal strategy for Player A has any active strategies except*:

$$\overline{B}_1 = 0,\ A_2 = n;\ \overline{B}_1 = 1,\ A_2 = n;\ \cdots;\ \overline{B}_1 = n,\ A_2 = n.$$

Proof for the three-coin game. These strategies have been shown to be active in one optimal strategy mix. We now prove that no other pure strategy can be active in an optimal mixed strategy for A in the three-coin game.

Let T be an A-strategy with $A_2\neq 3$. First consider the case $A_2<2$. We choose the following two-strategy mix for B: (1) S_2, a Type 2 strategy with -1 entries against T and every A-strategy for which $\overline{B}_1=3$, (2) S_3, a Type 3 strategy with -1 entries against every A-strategy for which $\overline{B}_1=2$. Other -1 entries are immaterial as long as condition 2 is satisfied. If these strategies are played with equal weights, a nonpositive expected payoff to A is assured for every pure A-strategy, and a negative payoff of $-1/2$ is obtained when strategy T is played in particular. Similarly, if $A_2>4$, a mix of a Type 0 and Type 1 strategy

suffices to provide a negative expected payoff.

It remains to consider the case $A_2=2$ or 4 (in general, $A_2=n\pm1$). Suppose $A_2=2$. For this case, we construct a four strategy optimal mix for B in a manner similar to that employed in the proof of Theorem 2. We begin by defining four strategies of Types 0, 1, 2, and 3, denoted S_0, S_1, S_2, and S_3, respectively. S_3 will be called the key strategy. Suppose strategy T is defined by $\overline{B}_1=i$, $A_2=2$. Choose an integer $j\leqq3$, $j\neq i$. In column S_j, insert a -1 against strategy T. This is possible since $\overline{B}_1(T)=i$, and $j\neq i$. Now in column S_i, find an A-strategy with $A_2=2$, $\overline{B}_1\neq i$, j. Such a strategy exists, say $\overline{B}_1=k$, since three occurrences of $A_2=2$ are shown in Table 4. Furthermore, $k\neq3$, since the A-strategy $\overline{B}_1=3$, $A_2=2$ is not feasible. In column S_k, assign a -1 against the A-strategy with $A_2=2$, $\overline{B}_1=j$. Also, in column S_i, assign a -1 against the A-strategy with $A_2=2$, $\overline{B}_1=k$.

Now enter a -1 in column S_3 (the key strategy) against T. This is the fourth -1 entry against A-strategies having $A_2=2$, and therefore, the second such entry against T. The remaining entries in column S_3 are entered in any manner to create a feasible strategy and satisfy Condition 2. The remaining entries in columns S_0, S_1, and S_2 are filled in the manner described in the proof of Theorem 2 (except for the -1's on lines where $A_2=2$ which have already been defined). The resulting strategy mix has one $+1$ and one or more -1 on every line, and in particular on line T, it has two -1's. Hence, when these strategies are played in equal proportion against any A pure strategy, the payoff to A is nonpositive, and against T in particular, it is negative and equal to $-1/4$.

This proves the theorem for the case $A_2=2$. For $A_2=4$, the proof is similar, except that S_0 is the key strategy. This completes the proof of the theorem.

Remark. It is clear from Theorem 3 and the proof of Theorem 1 that A must play these strategies with equal weights.

7. Concluding remarks. Despite the content of Theorem 1, it seems apparent that the advantage in the game lies with Player B wherever $n>1$. He has a wide selection of optimal strategy mixes, and in view of Theorems 2 and 3, he can select them in such a way that he can penalize A for any departure from the strategies in A's single optimal mix, while simultaneously protecting himself against any loss. Player A, on the other hand, is limited to his single mixed strategy, and may not deviate from it without risking loss. Unfortunately for A, while his strategy assures a nonnegative expected payoff, it also assures a nonpositive payoff, even if B blunders. While a quantitative measure of the value of such a property of a game is not generally agreed on [3], it is difficult to imagine any interpretation which would not make this circumstance a plus value for B.

It has been pointed out to the author that in view of this, a wiley A-player may choose a nonfeasible strategy such as $A_1=3$, $A_2=2$ in order to force a drawn play. In accordance with the convention mentioned in Section 2, this player would then attain on the next play the B-role, which is preferable. The inter-

pretation of this strategem will be left to the reader.

It will be noted that in the three-coin game, one half of the plays end in draws (provided both players play perfectly). This may be seen by examination of the proof of Theorems 1 and 3. The single feasible strategy mix which A will employ will win one play out of four against any B-strategy, and lose one out of four against any feasible B-strategy, and draw in the remaining two plays. In general, the ratio of draws to total plays is readily seen to be $(n-1)/(n+1)$.

An interesting consequence of the theory is that B may play the game with only one coin for any n. In the proof of Theorem 1, a construction is given which provides an optimal mixed strategy for B comprising Type 0 and 1 strategies only. The content of Theorems 2 and 3 must be modified, of course, if B has only one coin. However, the methods employed in the proof of the theorems may readily be applied to determine the new relationships.

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MATHEMATICAL NOTES

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NOTE ON STIRLING'S FORMULA

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Stirling's asymptotic formula, namely,

$$n! \sim \sqrt{(2\pi n)} \cdot (n/e)^n, \qquad n \to \infty,$$

is usually proved by showing that

$$(1) \qquad n! = \sqrt{(2\pi n)} \cdot (n/e)^n \cdot e^{\gamma_n}, \qquad 0 < \gamma_n < 1/(12n), \qquad n = 1, 2, \cdots.$$

Herbert Robbins (this MONTHLY, vol. 62, 1955, pp. 26–29) has shown by an elementary method that the estimate for γ_n in (1) can be replaced by the improved estimate

$$(2) \qquad 1/(12n+1) < \gamma_n < 1/(12n), \qquad n = 1, 2, \cdots.$$

In this note, we shall prove, by a simple refinement of the argument of Robbins, the stronger and more precise result

$$(3) \qquad n! = \sqrt{(2\pi n)} \cdot (n/e)^n \cdot \exp\left(\frac{1}{12n} - \frac{\theta_n}{360n^3}\right), \quad 0 < \theta_n < 1, \quad n = 1, 2, \cdots,$$

as envisaged by Euler's summation formula.

For the proof, let $a_n = n!n^{-1/2}(e/n)^n$, $l_n = \log a_n$, $n = 1, 2, \cdots$. We have

$$\frac{a_n}{a_{n+1}} = \frac{1}{e}\left(1 + \frac{1}{n}\right)^{n+1/2}, \qquad l_n - l_{n+1} = \left(n + \frac{1}{2}\right)\log\left(1 + \frac{1}{n}\right) - 1.$$

Using the familiar series

$$\log\left(1 + \frac{1}{n}\right) = \frac{2}{2n+1}\left(1 + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \cdots\right),$$

we have

$$(*) \qquad l_n - l_{n+1} = \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \cdots.$$

First we derive, as usual, from (*):

$$0 < l_n - l_{n+1} < \frac{1}{3(2n+1)^2}\left(1 - \frac{1}{(2n+1)^2}\right)^{-1} = \frac{1}{12n(n+1)}.$$

On rewriting this in the form $0 < l_n - l_{n+1} < (1/12)\{(1/n) - (1/(n+1))\}$, we get on the one hand

$$(4) \qquad l_n > l_{n+1}, \qquad n = 1, 2, \cdots$$

and, on the other,

$$(5) \qquad l_n - \frac{1}{12n} < l_{n+1} - \frac{1}{12(n+1)}, \qquad n = 1, 2, \cdots.$$

From (4) and (5) we obtain the known result that the sequence of intervals $I_n: l_n - (1/(12n)) < \lambda < l_n$, $n = 1, 2, \cdots$, whose lengths evidently tend to zero, forms a nest assuring existence of the limit

$$(6) \qquad \lambda = \lim_{n\to\infty} l_n.$$

This, in turn, assures the existence of the limit

$$(7) \qquad \alpha = \lim_{n\to\infty} a_n = e^{\lambda} > 0.$$

We use the known value

$$(8) \qquad \alpha = (2\pi)^{1/2}.$$

Indeed, (8) follows, after (7), on letting $n \to \infty$ in the easily verified relation

$$\frac{a_n^2}{a_{2n}} = \left(\frac{2}{n}\right)^{1/2} \frac{2\cdot 4\cdot \cdots \cdot (2n)}{1\cdot 3\cdot \cdots \cdot (2n-1)}$$

and using Wallis' well-known product for π.

We now proceed to improve (4), so as to lead to (3), by finding a suitable series majorized by (*). The series obtained by replacing the pth term in (*) by $3^{-p}(2n+1)^{-2p}$ for each p was employed by Robbins and led him to (2). It fails, however, to yield the more precise estimate we have in mind. On the other hand, if we retain the second term there and allow the others to be replaced as above, we still get a series majorized by (*) and it is interesting to find that it just yields our result. In fact, we have

$$\begin{aligned} l_n - l_{n+1} &> \frac{1}{3(2n+1)^2}\left(1 - \frac{1}{3(2n+1)^2}\right)^{-1} + \left(\frac{1}{5} - \frac{1}{3^2}\right)\frac{1}{(2n+1)^4} \\ &= \frac{1}{12\ \ n+1)}\left(1 + \frac{1}{6n(n+1)}\right)^{-1} + \frac{1}{180n^2(n+1)^2}\left(1 + \frac{1}{4n(n+1)}\right)^{-2} \\ &> \frac{1}{12n(n+1)}\left(1 - \frac{1}{6n(n+1)}\right) + \frac{1}{180n^2(n+1)^2}\left(1 - \frac{1}{2n(n+1)}\right) \\ &= \frac{1}{12n(n+1)} - \frac{3n(n+1)+1}{360n^3(n+1)^3}. \end{aligned}$$

As before this gives

$$l_n - l_{n+1} > \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{360}\left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right),$$

which shows that

$$(9) \qquad l_n - \frac{1}{12n} + \frac{1}{360n^3} > l_{n+1} - \frac{1}{12(n+1)} + \frac{1}{360(n+1)^3}, \qquad n = 1, 2, \cdots.$$

It follows from (5) and (9) that the sequence of intervals

$$J_n\colon\ l_n - \frac{1}{12n} < \lambda < l_n - \frac{1}{12n} + \frac{1}{360n^3}, \qquad n = 1, 2, \cdots,$$

whose lengths obviously tend to zero, forms a nest assuring the existence of the limit (6), since the endpoints of J_n differ from those of I_n by $o(1)$ as $n \to \infty$.

The result just proved yields what we desire. In fact, we may now write

$$(10) \qquad l_n = \lambda + \frac{1}{12n} - \frac{\theta_n}{360n^3}, \qquad 0 < \theta_n < 1, \qquad n = 1, 2, \cdots.$$

Recalling the definitions of a_n and l_n, (10) together with (7) and (8) clearly establishes (3) and the proof is complete.

ESTIMATION IN A CERTAIN PROBABILITY PROBLEM

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An example sometimes quoted as a "nonintuitive" probability is the following: Given a collection of K people, find the probability that some two of them were born on the same day of the year (assuming that no one was born on February 29, and that people are born with equal probability on other days). If K is 23, the probability slightly exceeds 1/2, and if K is 50, the odds are about 33 to 1 in favor of such a pair.

This is of course a special case of the following problem. Let an experiment with N equally likely outcomes be performed K times. What is the probability, $P(N, K)$, that at least one of the outcomes occurs twice? $P(N, K) = 1 - Q(N, K)$ where $Q(N, K) = N(N-1) \cdots (N-K+1)/N^K = N!/[(N-K)!N^K]$ is the probability that all K outcomes are distinct ($K < N$). For given N and K this may be evaluated directly, or approximated with the aid of Stirling's formula.

The problem becomes more difficult, however, if one assigns values to $P(N, K)$ and N, and attempts to solve for K. We are then confronted with an equation

$$1 - \frac{N!}{(N-K)!N^K} = t$$

or, if Stirling's formula is used,

$$\left(\frac{N}{N-K}\right)^{N-K+1/2} e^{-K} = 1 - t,$$

to be solved for K, where $t = P(N, K)$. Such transcendental equations can be solved approximately, of course, when N and t are specified, but to exhibit $K = f(N, t)$ explicitly seems rather difficult. It is the purpose of this note to show that for $0 < t < 1$, K is given asymptotically by $L(t)\sqrt{N}$, where $L(t) = \sqrt{-2 \log (1-t)}$. Furthermore, except for extreme values of t, this approximation is very good even for small N.

We first prove the

LEMMA. *If $Q(N, K) \geqq a$, where a is constant, $0 < a < 1$, then $K/N \to 0$ as $N \to \infty$.*

*Proof.** Since $1 - x < e^{-x}$ for all x, we have

$$a \leqq Q(N, K) = \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{K-1}{N}\right)$$

$$\leqq \exp\left[-\left(\frac{1}{N} + \cdots + \frac{K-1}{N}\right)\right] = \exp\left[-\frac{K(K-1)}{2N}\right],$$

so that $K(K-1) \leqq 2N \log (1/a)$, which implies the conclusion of the lemma.

* The author is indebted to the referee for suggesting this simpler proof of the lemma.

THEOREM. *Let K, N, and t be related by the equation*

$$1 - \frac{N!}{(N-K)!N^K} = t.$$

(We may assume all three are continuous real variables, $0<K<N$.) Then

$$\frac{K^2}{-2N\log(1-t)} \to 1 \text{ as } N \to \infty.$$

Thus K is given asymptotically by $K = L(t)\sqrt{N}$, where $L(t) = \sqrt{-2\log(1-t)}$.

Proof. By Stirling's inequality we have

$$\left(\frac{N}{N-K}\right)^{N-K+1/2} e^{-K}\left(\frac{12(N-K)-1}{12(N-K)}\right) < \frac{N!}{(N-K)!N^K} < \left(\frac{N}{N-K}\right)^{N-K+1/2} e^{-K}\frac{12N}{(12N-1)}.$$

Since the middle member above is $1-t$, we have, by inverting,

$$(1-K/N)^{N-K+1/2}e^K = 1/[(1-t)(1+\epsilon)],$$

where $\epsilon \to 0$ as $N \to \infty$ (since $N-K \to \infty$ when $N \to \infty$).

Taking logarithms gives

$$\begin{aligned}
-\log[(1-t)(1+\epsilon)] &= K + (N-K+\tfrac{1}{2})\log(1-K/N) \\
&= K - (N-K)\sum_{r=1}^{\infty}\frac{1}{r}\frac{K^r}{N^r} + \frac{1}{2}\log(1-K/N) \\
&= K - K - \sum_{r=2}^{\infty}\frac{1}{r}\frac{K^r}{N^{r-1}} + \sum_{r=1}^{\infty}\frac{1}{r}\frac{K^{r+1}}{N^r} + \frac{1}{2}\log(1-K/N) \\
&= \frac{1}{2}\log(1-K/N) + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)\frac{K^{n+2}}{N^{n+1}} \\
&= \frac{1}{2}\log(1-K/N) + \frac{K^2}{2N}\left[1 + \sum_{n=1}^{\infty}\frac{2}{(n+1)(n+2)}\frac{K^n}{N^n}\right].
\end{aligned}$$

The lemma shows that $K/N \to 0$ as $N \to \infty$, so we have $-\log[(1-t)(1+\epsilon)] = \epsilon' + [K^2/(2N)](1+\epsilon'')$, where $\epsilon' = \frac{1}{2}\log(1-K/N) \to 0$ as $N \to \infty$ and

$$0 < \epsilon'' = \sum_{n=1}^{\infty}\frac{2}{(n+1)(n+2)}\frac{K^n}{N^n} < \sum_{n=1}^{\infty}\frac{K^n}{N^n} = \frac{K/N}{1-K/N} \to 0$$

as $N \to \infty$. Thus

$$\frac{K^2}{2N} = \frac{-\log[(1-t)(1+\epsilon)] - \epsilon''}{1+\epsilon'},$$

so that

$$\frac{K^2}{[L(t)]^2N} \quad \frac{-\log[(1-t)(1+\epsilon)]-\epsilon''}{-(1+\epsilon')\log(1-t)} \to 1.$$

It is of some interest to see how good the approximation is for several values of N. The author has done this with the aid of the Univac 1103A and some of the results are tabulated below. In this table, K is the least integer such that $N!/[(N-K)!N^K] \leqq 1-t$ and K_1 is the integer nearest $L(t)\sqrt{N}$.

	$t = \frac{1}{32}$		$t = \frac{1}{4}$		$t = \frac{1}{2}$		$t = \frac{3}{4}$		$t = 31/32$	
N	K	K_1	K	K_1	K	K_1	K	K_1	K	K_1
4	2	1	2	2	3	2	4	3	5	5
4^2	2	1	4	3	5	5	7	7	10	11
4^3	3	2	7	6	10	9	14	13	21	21
4^4	5	4	13	12	20	19	27	27	42	42
4^5	9	8	25	24	38	38	54	53	84	84
4^6	17	16	49	49	76	75	107	107	168	168
4^7	33	32	98	97	151	151	214	213	337	337
4^8	65	65	195	194	302	301	427	426	674	674
4^9	130	129	389	388	604	603	853	853	1348	1348
4^{10}	259	258	778	777	1206	1206	1706	1705	2696	2696
4^{11}	517	516	1554	1553	2412	2411	3411	3410	5392	5392
4^{12}	1033	1032	3108	3107	4823	4823	6821	6820	10784	10784
4^{13}	2065	2064	6215	6214	9646	9645	13641	13641	21567	21568
4^{14}	4130	4129	12429	12428	19291	19291	27282	27281	43135	43135
4^{15}	8258	8257	24856	24855	38582	38581	54563	54562	86270	86271
4^{16}	16515	16514	49712	49711	77164	77163	109125	109125	172541	172541
4^{17}	33029	33028	99423	99422	154326	154325	218250	218249	345082	345082

AN ANALYTICAL EXPRESSION FOR $[X]$

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THEOREM. *Given the functions*

$$F(X) = \frac{\operatorname{Arcsin}|\sin \pi x|}{\pi} \quad \textit{and} \quad G(X) = \lim_{N\to\infty} \{1 + |\sin \pi(X - F(X))|\}^N,$$

then for all finite X, $[X]$ *(the greatest integer not greater than* X*) may be written as*

$$[X] = X - |F(X) + 2^{1-G(X)} - 1|.$$

To prove the theorem it is helpful to establish two lemmas.

LEMMA 1. *For* K *an integer and* $0 \leqq b \leqq \frac{1}{2}$, $F(K+b) = b$; *for* $\frac{1}{2} < b < 1$, $F(K+b) = 1-b$.

LEMMA 2. *For* K *an integer and* $0 \leqq b \leqq \frac{1}{2}$, $G(K+b) = 1$; *for* $\frac{1}{2} < b < 1$, $G(K+b) = \infty$.

To prove Lemma 1, compute $F(K+b)$, where $0 \leqq b \leqq \frac{1}{2}$. Then

$$F(K+b) = \frac{\text{Arcsin}\,|\sin(K\pi + b\pi)|}{\pi} = \frac{\text{Arcsin}\,|\sin(b\pi)|}{\pi} = b.$$

Now consider $F(K+b)$, where $\frac{1}{2} < b < 1$. We have

$$F(K+b) = \frac{\text{Arcsin}\,|\sin(K\pi + b\pi)|}{\pi} = \frac{\text{Arcsin}\,|\sin(b\pi)|}{\pi}$$

$$= \frac{\pi - b\pi}{\pi} = 1 - b.$$

The proof of Lemma 2 is also direct. Consider $G(K+b)$, where $0 \leqq b \leqq \frac{1}{2}$. Then $G(K+b) = \lim_{N\to\infty}\{1+|\sin(K+b-b)\pi|\}^N = \lim_{N\to\infty}\{1+0\}^N = 1$, where Lemma 1 is used to prove that in this case $X - F(X)$ is the integer K. Then consider $G(K+b)$ for $\frac{1}{2} < b < 1$. We have $G(K+B) = \lim_{N\to\infty}\{1+|\sin(K-1+2b)\pi|\}^N = \lim_{N\to\infty}\{1+|\sin(2b\pi)|\}^N = \infty$, since $\sin(2b\pi) \neq 0$ for the assumed range of b.

It is now possible, with the aid of the lemmas, to prove the theorem. We assume that $X = K+b$, where $0 \leqq b < 1$, so that $[X] = K$. Two cases are considered.

Case 1. $0 \leqq b \leqq \frac{1}{2}$. Substituting $X = K+b$ in the formula of the theorem, we find $[X] = K+b-|b+2^{1-1}-1| = K$, where each step is justified by one of the lemmas.

Case 2. $\frac{1}{2} < b < 1$. Letting $X = K+b$ in the formula of the theorem, we have $[X] = K+b-|1-b+2^{1-\infty}-1| = K+b-|-b| = K$.

These two cases prove the theorem.

Since the cost, in cents, for a letter of N ounces may be written $C = -A[-N]$, where A is the cost per ounce, it is possible to write an analytical formula for this "postal function."

The writer wishes to express his appreciation for some very helpful suggestions on the part of the referee.

ANOTHER PROOF FOR TWO PROPERTIES OF THE GENERATORS OF A RULED SURFACE

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C. E. Weatherburn [2] has proved the following theorem for ruled surfaces, referred to the generators v = constant and their orthogonal trajectories.

When $\mathbf{a}$ is the unit tangent vector of the generators, so that div $\mathbf{a}$ is equal to the geodesic curvature of the orthogonal trajectories u = constant, ψ is the distance function for the family of generators and K the Gaussian curvature, then

$$(1) \qquad \frac{\partial}{\partial u}\sqrt{(-K)} = \frac{\partial}{\partial u}\left(\frac{M}{\psi}\right) = M_\psi \frac{\partial}{\partial u}\left(\frac{1}{\psi^2}\right) = -2\sqrt{(-K)}\ \text{div}\,\mathbf{a}.$$

This theorem is obtained by using the Mainardi-Codazzi equations. Ram Behari [1] has proved it without reference to these equations. An alternative proof, which only depends on the material in [2], Sections 29, 30 is as follows:

Suppose u denotes the actual distance along a generator; then we have $E=1$, $F=0$. Since the normal curvature in the direction of a generator is zero, it follows that $L=0$, so that the first and second curvature are given by:

$$J = N/G, \tag{2}$$

$$K = -M^2/G. \tag{3}$$

The distance function ψ for the family of generators $v=$constant is the reciprocal of the magnitude of ∇v. Consequently,

$$\psi = 1/|\nabla v| = 1/\sqrt{(|\mathbf{r}_2|^2/G^2)} = \sqrt{G}, \tag{4}$$

and

$$\operatorname{div} \mathbf{a} = G_1/(2G) = \psi_1/\psi. \tag{5}$$

Since $\psi\psi_{11}+\psi_1^2=1$, integration with respect to u gives $\psi\psi_1=u-\alpha$, and therefore $\psi^2=(u-\alpha)^2+\beta^2$, where α and β are functions of v only. Now

$$K = -\beta^2/\psi^4. \tag{6}$$

On the basis of (5) and (6), we obtain immediately

$$\frac{\partial}{\partial u}\sqrt{(-K)} = \frac{-2\beta\psi\psi_1}{\psi^4}$$

and therefore that

$$\frac{\partial}{\partial u}\log\sqrt{(-K)} = \frac{-2\psi\psi_1}{\psi^2} = -2\operatorname{div}\mathbf{a}.$$

That is,

$$\frac{\partial}{\partial u}\sqrt{(-K)} = -2\sqrt{(-K)}\operatorname{div}\mathbf{a}.$$

Ram Behari [1] also proved, now with the aid of the Mainardi-Codazzi equations, the following theorem for a ruled surface using an orthogonal coordinate system with $v=$constant as asymptotic lines, so that $L=0$, $F=0$.

If the surface is ruled and the mean curvature varies inversely as the distance function along the generator, then the total curvature is constant along any orthogonal trajectory of the generators.

Here the derivation is somewhat long and complicated. We point out that the result can be derived directly: Since ([2], Sec. 29)

$$\frac{\partial}{\partial v}\left(\frac{M}{\psi}\right)=\frac{\partial}{\partial u}\left(\frac{N}{\psi}\right), \qquad \frac{\partial}{\partial u}(M\psi)=0,$$

for $M\neq 0$,

$$\frac{\partial}{\partial v}\sqrt{(-K)}=\frac{\partial}{\partial v}\left(\frac{M}{\psi}\right)=\frac{\partial}{\partial u}\left(\frac{MN}{M\psi}\right)=\frac{1}{M\psi}\frac{\partial}{\partial u}(\sqrt{G}\cdot J\cdot G\sqrt{(-K)})$$

$$=\frac{1}{M\psi}\left[\psi^2\sqrt{(-K)}\frac{\partial}{\partial u}(\psi J)\right]=\frac{\psi}{M}\sqrt{(-K)}\frac{\partial}{\partial u}(\psi J).$$

Consequently we have

$$\frac{\partial}{\partial v}\sqrt{(-K)}=\frac{\partial}{\partial u}(\psi J).$$

from which the theorem follows.

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CLASSROOM NOTES

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THE HIGHER DERIVATIVE TEST FOR EXTREMA AND POINTS OF INFLECTION

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1. Introduction. Consider a real function f of one real variable x. When the second derivative as well as the first vanishes at a point P: $(x_0, f(x_0))$, the second derivative test for determining whether P is an extremum of f fails to apply. Under these circumstances more advanced books on calculus give a higher derivative test for determining whether P is an extremum [1–9]. The proof always seems to use a higher order mean value theorem, that is, Taylor's theorem. Usually the first of the higher derivatives which does not vanish at x_0, say $f^{(s)}$, is required to be continuous in an open interval (a, b) which includes x_0 [1–7]. However, the hypotheses of the test can be weakened: The existence and be-

havior of $f^{(s)}$ at any point but x_0 is irrelevant; $f^{(s-1)}$ must exist throughout some open interval (a, b) including x_0, but need not be continuous there. Occasionally the test is proved in this more general form by applying Taylor's theorem to $f^{(s-1)}$, then using some extra argument [8, 9]. The present note proves the more general form of the test without using Taylor's theorem, which the author feels is too sophisticated for the average first year student of calculus.

2. Examples. Before beginning the proof we show that the extension of the higher derivative test is not trivial by finding functions whose sth derivative exists at x_0 but is discontinuous there. To these functions the extended test can still be applied at x_0, although the usual test can not. Consider

$$\phi_r(x) = x^r \cos(1/x),\ x \neq 0;\ \phi_r(0) = 0;$$

where $r=\rho/(2\sigma+1)>0$ and ρ, σ are integers. It is not too hard to show that

$$\phi_r^{(s)}(x) = x^{r-2s}\cos\left(\frac{1}{x}-\frac{\pi s}{2}\right) - A_{rs}x^{r-2s+1}\sin\left(\frac{1}{x}-\frac{\pi s}{2}\right) + O(x^{r-2s+2}) \quad \text{for } x \neq 0,$$

$$\phi_r^{(s)}(0) = 0 \qquad \text{for } r > 2s-1,$$

where $A_{rs}>0$ depends only on r and s, and O denotes the order of the other terms as x goes to 0. It follows that $\phi_r^{(s)}$ is defined at 0 but discontinuous there whenever $2s-1<r\leqq 2s$. When $r<2s$, the oscillations of $\phi_r^{(s)}$ tend to infinity as x tends to 0.

Consider now the functions

$$f(x) = x^{2n} + x^{4n-\delta}\cos(1/x), \qquad x \neq 0; \qquad f(0) = 0;$$
$$g(x) = x^{2n+1} + x^{4n+2-\delta}\cos(1/x), \qquad x \neq 0; \qquad g(0) = 0;$$

where $0\leqq\delta=\alpha/(2\beta+1)<1$ and α, β are integers. Then $f'(0)=f''(0)=\cdots=f^{(2n-1)}(0)=0$ but $f^{(2n)}(0)=(2n)!$, and $g'(0)=g''(0)=\cdots=g^{(2n)}(0)=0$ but $g^{(2n+1)}(0)=(2n+1)!$, while neither $f^{(2n)}$ nor $g^{(2n+1)}$ is continuous at 0. By Theorem 1 below f has a minimum at 0 and g has a point of inflection there. Thus Theorem 1 is a nontrivially stronger form of the higher derivative test for extrema and points of inflection.

Two other examples may be of interest. Let ψ_0 be an everywhere continuous, nowhere differentiable function, such as that of Weierstrass. Let $\psi_n(x) = \int_0^x \psi_{n-1}(z)dz$ for $n=1, 2, \cdots$. Consider $h(x)=x\psi_{s-1}(x)$. Then $h^{(j)}(x)=x\psi_{s-j-1}(x) + j\psi_{s-j}(x)$ for $j=1, \cdots, s-1$, while $h^{(s)}(0)=s\psi_0(0)$ but $h^{(s)}(x)$ does not exist for $x\neq 0$.

As a last example let u be defined by

$$u(x) = \left\{\begin{array}{ll} 0 & \text{for } x = 0 \\ x^{1+r} & \text{for } 2^{-(2n+1)} < |x| \leqq 2^{-2n} \\ -x^{1+r} & \text{for } 2^{-2n} < |x| \leqq 2^{-(2n-1)} \end{array}\right\} \quad \text{for integer } n,$$

where $r=\rho/(2\sigma+1)>0$ and ρ, σ are integers. Then $u'(x)$ is defined except when $x=2^{-n}$, and $u'(0)=0$. It follows that neither u nor u' is continuous in any open interval (a, b) which includes 0. The existence of its derivative at a point ensures the continuity of a function at the point, but not its continuity in any open interval including that point.

3. The higher derivative test. The higher derivative tests can be stated as follows:

THEOREM 1. *Let n denote a positive whole number, f a real function of one real variable.*

A. *If $f'(x_0)=f''(x_0)=\cdots=f^{(2n-1)}(x_0)=0$, then:*

(1) *$f^{(2n)}(x_0)>0$ implies that $f(x_0)$ is a minimum value of f and that f is concave upwards at the point $(x_0, f(x_0))$;*

(2) *$f^{(2n)}(x_0)<0$ implies that $f(x_0)$ is a maximum value of f and that f is concave downwards at the point $(x_0, f(x_0))$.*

B. *If $f''(x_0)=f'''(x_0)=\cdots=f^{(2n)}(x_0)=0$, then: $f^{(2n+1)}(x_0)\neq 0$ implies that $f(x_0)$ is not an extreme value of f, even if $f'(x_0)=0$, but that $(x_0, f(x_0))$ is a point of inflection of f.*

(1) *If $f^{(2n+1)}(x_0)>0$, then f is concave downwards to the left, concave upwards to the right of x_0.*

(2) *If $f^{(2n+1)}(x_0)<0$, then f is concave upwards to the left, concave downwards to the right of x_0.*

(3) *If in addition $f'(x_0)=0$, then $f^{(2n+1)}(x_0)>0$ implies that f is increasing, and $f^{(2n+1)}(x_0)<0$ implies that f is decreasing at the point $(x_0, f(x_0))$.*

First we prove Theorem 1-A for $n=1$. We assume as already known the following theorems [10].

THEOREM 2. *If g is a function and $g'(x_0)\neq 0$ exists, then there is an open interval (a, b) including x_0 such that $g(x)$ exists for every x within the interval and $[g(x)-g(x_0)]/(x-x_0)$ maintains the same sign as $g'(x_0)$ for every such x but x_0.*

If $g'(x_0)>0$, then $g(u)<g(x_0)<g(v)$ whenever $a<u<x_0<v<b$.

If $g'(x_0)<0$, then $g(u)>g(x_0)>g(v)$ whenever $a<u<x_0<v<b$.

In applying Theorem 2 we shall consistently take (a, b) to be the most inclusive open interval satisfying the conditions of that theorem.

THEOREM 3. *If $g'(x)$ exists for every x in an open interval (a, b) and maintains the same sign throughout, then:*

(1) *$g'(x)>0$ for every x in (a, b) implies that g is strictly increasing there and that $g(a+)<g(u)<g(v)<g(b-)$ whenever $a<u<v<b$;*

(2) *$g'(x)<0$ for every x in (a, b) implies that g is strictly decreasing there and that $g(a+)>g(u)>g(v)>g(b-)$ whenever $a<u<v<b$.*

Suppose then that $f'(x_0)=0$ and $f''(x_0)>0$. Then, applying Theorem 2 to the function f', $f'(u)<0<f'(v)$ whenever $a<u<x_0<v<b$. It follows from Theorem 3

that $f(a+)>f(u)>f(x_0)$ and $f(x_0)<f(v)<f(b-)$, that is, $f(x)>f(x_0)$, where $x\neq x_0$ is any other point in (a, b), and $f(x_0)$ is a local minimum value of f. For every x in (a, b) but x_0, $f(x)$ lies above the line $y=f(x_0)$, which is the tangent line to the graph of f at the point $(x_0, f(x_0))$. Hence f is concave upwards at x_0. Similarly, if $f'(x_0)=0$ and $f''(x_0)<0$, one can prove the second half of Theorem 1-A for $n=1$. Theorem 1-A with $n=1$ is commonly called the second derivative test for extrema.

We now prove Theorem 1-A for general n. Suppose that $f^{(2n-1)}(x_0)=0$ and $f^{(2n)}(x_0)>0$. Then, applying Theorem 2 to the function $f^{(2n-1)}$, one finds that $f^{(2n-1)}(x)$ exists for every x in (a, b), the open interval, including x_0, where $[f^{(2n-1)}(x)-f^{(2n-1)}(x_0)]/(x-x_0)>0$ for every x but x_0. The second derivative test is applied to the function $f^{(2n-2)}$ and one finds that $f^{(2n-2)}(x)>f^{(2n-2)}(x_0)=0$, where $x\neq x_0$ is any other point in (a, b). The same line of argument is used when $f^{(2n-1)}(x_0)=0$ and $f^{(2n)}(x_0)<0$. The existence of $f^{(2n-1)}(x)$ for every point x in (a, b) ensures the continuity there of f and its derivatives of order less than $2n-1$.

The second derivative test can not be used further, for by hypothesis $f^{(2r)}(x_0)=0$ for $r=1, \cdots, n-1$. To continue the proof we establish the following lemma.

LEMMA 4. *Let h be a function for which $h''(x)$ exists for every x in an open interval (a, b), and let $h'(x_0)=h''(x_0)=0$ for x_0 some point of (a, b).*

(1) If $h''(x)>0$ for every x in (a, b) except x_0, then, within (a, b), h' is strictly increasing, $h(x_0)$ is the minimum value of h, and h is concave upwards.

(2) If $h''(x)<0$ for every x in (a, b) except x_0, then, within (a, b), h' is strictly decreasing, $h(x_0)$ is the maximum value of h, and h is concave downwards.

Suppose that $h'(x_0)=h''(x_0)=0$ and that $h''(x)>0$ for every other x in (a, b) except x_0. Then, applying Theorem 3 to the function h', h' is strictly increasing in the open intervals (a, x_0) and (x_0, b) and $h'(a+)<h'(u)<h'(x_0-)=0=h'(x_0+)<h'(v)<h'(b-)$ whenever $a<u<x_0<v<b$. Thus h' is strictly increasing throughout the open interval (a, b). Since $h'(u)<0<h'(v)$, it follows from Theorem 3 that $h(a+)>h(u)>h(x_0)$ and $h(x_0)<h(v)<h(b-)$, that is, $h(x)>h(x_0)$, where $x\neq x_0$ is any other point in (a, b), and $h(x_0)$ is a local minimum value of h. For every x in (a, b) but x_0, $h(x)$ lies above the line $y=h(x_0)$, which is the tangent line to the graph of h at the point $(x_0, h(x_0))$. Hence h is concave upwards at x_0. One can readily prove that h is in fact concave upwards for every x in (a, b). However, we do not need this to establish our main result and omit the proof of this detail. Similarly if $h'(x_0)=h''(x_0)=0$ and if $h''(x)<0$ for every other x in (a, b) except x_0, one can prove the second half of Lemma 4.

We return now to the proof of Theorem 1-A for general n. Suppose we have established that $f^{(2r)}$ is continuous throughout the open interval (a, b) and that $f^{(2r)}(x)>f^{(2r)}(x_0)$ whenever $x\neq x_0$. By hypothesis $f^{(2r-1)}(x_0)=f^{(2r)}(x_0)=0$. Then by Lemma 4, $f^{(2r-1)}$ is strictly increasing throughout (a, b), $f^{(2r-1)}(u)<0<f^{(2r-1)}(v)$ whenever $a<u<x_0<v<b$, $f^{(2r-2)}(x_0)$ is the minimum value of $f^{(2r-2)}$ within

(a, b); and $f^{(2r-2)}$ is concave upwards throughout (a, b). Again, $f^{(2r-2)}$ is continuous throughout (a, b), $f^{(2r-2)}(x) > f^{(2r-2)}(x_0)$ there whenever $x \neq x_0$, and by hypothesis $f^{(2r-3)}(x_0) = f^{(2r-2)}(x_0) = 0$. If this argument can be started for some integer $r = s$, it can be continued for smaller integers until we reach $r = 1$. But we earlier established the proper conditions for this argument for $r = n-1$. At the $(n-1)$st step, with $r = 1$, we find that f' is strictly increasing throughout (a, b), $f'(u) < 0 < f'(v)$ whenever $a < u < x_0 < v < b$, and conclude that $f(x_0)$ is the minimum value of f within (a, b), and f is concave upwards throughout (a, b). Thus the first half of Theorem 1-A is established for general n. The same line of argument is used to prove the second half of Theorem 1-A for general n.

To prove Theorem 1-B apply Theorem 1-A to the function f'. Suppose that the other proper hypotheses hold and that $f^{(2n+1)}(x_0) > 0$. Then $f''(u) < 0 < f''(v)$ whenever $a < u < x_0 < v < b$, as we found in the proof of Theorem 1-A, both when $n = 1$ and in the general case. Thus f has a point of inflection at the point $(x_0, f(x_0))$, and f is concave downwards in the open interval (a, x_0), concave upwards in the open interval (x_0, b). Furthermore $f'(x_0)$ is the minimum value of f' in (a, b), and f' is continuous throughout this interval. If $f'(x_0) \geqq 0$, then $f'(x) > 0$ for every other $x \neq x_0$ in (a, b). Then f' does not change sign in (a, b), so $f(x_0)$ is not an extreme of f, but rather f is strictly increasing throughout (a, b). In particular this is true for $f'(x_0) = 0$, proving the first half of 1-B-3. On the other hand, if $f'(x_0) < 0$, then by Theorem 2 there is an open interval (c, d), possibly a subinterval of (a, b), where $[f(x) - f(x_0)]/(x - x_0) < 0$ for every x in (c, d) but x_0. Then $f(u) > f(x_0) > f(v)$ whenever $c < u < x_0 < v < d$, and again $f(x_0)$ is not an extreme value of f. Thus 1-B-1 is established. Similarly, with the other proper hypotheses and with $f^{(2n+1)}(x_0) < 0$, one can prove 1-B-2 and the second half of 1-B-3.

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A PROPERTY OF THE BERNOULLI NUMBERS

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The Bernoulli numbers in the even suffix notation satisfy the symbolic relation

$$(B+1)^n - B^n = 0 \qquad (n > 1), \tag{1}$$

where, after expansion, exponents are degraded to subscripts. By the Staudt-Clausen theorem, the denominator of B_{2n} contains the prime 2 to the first power. Thus both numerator and denominator of $2B_{2n}$ are odd after cancellation of 2. The more precise statement

$$2B_{2n} \equiv 1 \pmod 4 \qquad (n \geqq 2) \tag{2}$$

seems not to be very well known. It is implicit in a stronger result proved in Nielsen's book ([2] p. 256).

The congruence (2) can be proved rapidly as follows. From (1) we have

$$\sum_{r=0}^{2n} \binom{2n+1}{r} B_r = 0 \qquad (n > 0),$$

so that

$$2 - (2n+1) + \frac{1}{3}\binom{2n+1}{2} + \sum_{r=2}^{n-1} \binom{2n+1}{2r} 2B_{2r} + (2n+1)2B_{2n} = 0.$$

Assuming that (2) holds up to and including the value $n-1$, we have

$$2 - (2n+1) - \binom{2n+1}{2} + \sum_{2}^{n-1} \binom{2n+1}{2r} + (2n+1)2B_{2n} \equiv 0 \pmod 4.$$

But

$$\sum_{0}^{n} \binom{2n+r}{2r} = \frac{1}{2}\left\{(1+1)^{2n+1} + (1-1)^{2n+1}\right\} \equiv 0 \pmod 4$$

for $n \geqq 1$, so that

$$(2n+1)2B_{2n} + 1 - 2(2n+1) - 2\binom{2n+1}{2} \equiv 0,$$

$$(2n+1)(2B_{2n} - 1) \equiv 2n + 2\binom{2n+1}{2} \equiv 2n(2n+2) \equiv 0 \pmod 4.$$

This evidently proves (2).

By the same method we can prove that

$$2B_{2n} \equiv 4n + 1 \pmod 8 \qquad (n \geqq 2), \tag{3}$$

which is also implicit in a result given by Nielsen that is ascribed to Staudt. Indeed, since $4n+1\equiv(-1)^n 2-1 \pmod 8$, we have as above

$$2-(2n+1)+3\binom{2n+1}{2}+\sum_{2}^{n-1}(4r+1)\binom{2n+1}{2r}+(2n+1)2B_{2n} \equiv 0 \pmod 8.$$

But

$$\begin{aligned}\sum_{0}^{n}(4r+1)\binom{2n+1}{2r} &= 2(2n+1)\sum_{0}^{n}\binom{2n}{2r-1}+\sum_{0}^{n}\binom{2n+1}{2r}\\ &= (2n+1)\{(1+1)^{2n}-(1-1)^{2n}\}+\tfrac{1}{2}\{(1+1)^{2n+1}+(1-1)^{2n+1}\}\\ &\equiv 0 \pmod 8\end{aligned}$$

for $n\geqq 2$, so that

$$\sum_{2}^{n-1}(4r+1)\binom{2n+1}{2r}\equiv -1-5\binom{2n+1}{2}-(4n+1)(2n+1),$$

$$(2n+1)2B_{2n}+1-(2n+1)-2\binom{2n+1}{2}-(4n+1)(2n+1)\equiv 0,$$

$$(2n+1)(2B_{2n}-4n-1)\equiv 0 \pmod 8.$$

We recall that the numbers D_{2n} defined by

$$\frac{x}{\sinh x}=\sum_{0}^{\infty}D_{2n}\frac{x^{2n}}{(2n)!}$$

satisfy ([3] p. 28) $D_{2n}=2(1-2^{2n-1})B_{2n}$. Thus (2) implies

$$D_{2n}\equiv 1 \pmod 4, \tag{4}$$

while (3) implies

$$D_{2n}\equiv 4n+1 \pmod 8; \tag{5}$$

both (4) and (5) hold for all $n\geqq 0$.

We remark that if 2^r is the highest power of 2 dividing $2n$, then the writer has proved ([1] formula (10)) that

$$2B_{2n}\equiv 1 \pmod{2^{r+1}} \qquad (n\geqq 2); \tag{6}$$

this result is evidently in agreement with (3).

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ON TWO FUNCTIONAL EQUATIONS*

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It is well known that the only real-valued continuous solutions of the functional equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

are of the form $f(x) = kx$ where k is a real number. In this note we shall solve two functional equations which may be given to a beginning analysis class as applications of (1). Both of these equations arise naturally in describing the one-parameter groups of affine transformations in the euclidean plane.

THEOREM 1. *The only continuous solutions of the equation*

$$f(x+y) = f(x) + f(y) + a(1-A^x)(1-A^y), \tag{2}$$

where a and A are real numbers and $A>0$, are of the form

$$f(x) = kx - a(1-A^x) \tag{3}$$

where k is a real number.

Proof. Let us first find differentiable solutions of (2). Holding y constant and differentiating, we get

$$f'(x+y) = f'(x) + a(1-A^y)(-A^x \ln A).$$

Now holding x constant and differentiating, we obtain $f''(x+y) = a(\ln A)^2 A^{x+y}$. Let $u=x+y$; then $f''(u) = a(\ln A)^2 A^u$. $f'(u) = \int_0^u a(\ln A)^2 A^t dt = a(\ln A)[A^t]_0^u = a(\ln A)(A^u-1)$ and $f(u) = a\int_0^u (\ln A)(A^t-1)dt = a[A^t - t(\ln A)]_0^u$. Therefore, $f(u) = aA^u - au(\ln A) - a$. Let f_1 be any other continuous solution of (2); then $g=f-f_1$ is also continuous. We notice that

$$g(x+y) = (f-f_1)(x+y) = f(x) - f_1(x) + f(y) - f_1(y).$$

This relation shows that g is of form (1) and, therefore, there exists a real number c such that $g(x) = cx$. Hence, any continuous solution of (2) is of form (3).

THEOREM 2. *The only continuous solutions of $f(x+y) = A^x f(y) + A^y f(x)$, where A is a positive real number, are of the form $f(x) = kxA^x$ where k is a real number.*

Proof. Let $g(x) = A^{-x} f(x)$ and observe that g satisfies (1). Hence, there exists a real number k such that $g(x) = kx$. Therefore, $f(x) = kxA^x$.

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ON THE SYNTHETIC DIVISION PROCESS

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The authors present here a pair of proofs of a familiar result from college algebra. Stated without proof in many reputable texts, the theorem, with either of the proofs here given, has proved generally palatable to those freshmen conversant with the axiom of mathematical induction. The content of the theorem is, very roughly, that if nonintegral entries arise in the standard synthetic division process, then the rational number being tested is not a root of the equation in question. The result may be stated precisely as follows:

THEOREM. *Let p/q be a nonzero rational root of the polynomial equation $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where each a_k is an integer and $(p, q) = 1$. Let $b_0 = a_0$, and, for each positive integer k not exceeding n, let*

$$b_k = a_0(p/q)^k + a_1(p/q)^{k-1} + \cdots + a_k.$$

Then each b_k is an integer.

First proof. Assume that some $b_k = r/s$ is nonintegral, where $(r, s) = 1$. From the fact that

$$b_k = [a_0p^k + a_1p^{k-1}q + \cdots + a_kq^k]/q^k,$$

it follows that s divides q^k, so that $(p, s) = 1$. Then $(pr, s) = 1$, so that $(pr)/(qs) = (p/q)b_k$ is nonintegral. Then $b_{k+1} = (p/q)b_k + a_{k+1}$ is nonintegral.

$(n-k)$-fold repetition of this argument shows that b_n is nonintegral; in particular, b_n is nonzero.

Second proof. In this proof we show that each of the numbers b_k is an integer divisible by q.

That q divides the integer $b_0 = a_0$ is well known. Now let $0 \leqq k < n$, and suppose that b_k is an integer divisible by q. Then $b_{k+1} = (p/q)b_k + a_{k+1}$ is an integer which, in case $k+1 = n$, is zero and hence is divisible by q. To show that q divides b_{k+1} in case $k+1 < n$, set $x = p/q$ in the equation $f(x) = 0$ and multiply by q^{n-k-1} to get

$$a_0p^n/q^{k+1} + a_1p^{n-1}/q^k + \cdots + a_{k+1}p^{n-k-1} = -q[a_{k+2}p^{n-k-2} + \cdots + a_{n-1}pq^{n-k-3} + a_nq^{n-k-2}].$$

Since q divides the right-hand side, it divides the left-hand side, which is precisely

$$b_kp^{n-k}/q + a_{k+1}p^{n-k-1} = p^{n-k-1}[b_kp/q + a_{k+1}] = p^{n-k-1}b_{k+1}.$$

Since $(p, q) = 1$, it follows that q divides b_{k+1}.

AN APPLICATION OF STIRLING'S FORMULA

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Stirling's formula states that $n!\sim(2\pi n)^{1/2}n^ne^{-n}$. Varying only in certain refinements, the majority of proofs for Stirling's formula use the relation

$$\int_a^b \log x\,dx = b\log b - a\log a - b + a. \tag{1}$$

For our purpose (given below), it suffices to know that there exists elementary proofs which, *without using* (1), establish Stirling's formula in a weaker form, *i.e.*,

$$n! \sim Cn^{n+1/2}e^{-n}, \tag{2}$$

where C is an unspecified positive constant. Feller [1], Aissen [2], (there may be others) have given proofs of (2). The purpose of this note is to show how Stirling's formula, (2), may be used to establish the validity of (1) for real numbers, $b>a>0$.

We recall that a continuous function is Riemann-integrable over a closed interval. For $a>0$, $\log x$ is integrable on $a\leqq x\leqq b<\infty$. Thus, if we divide the interval $[a,\ b]$ into n equal parts such that $\Delta x_k=(b-a)/n$, $x_k=a+k\Delta x_k$, $k=0,\ 1,\ \cdots,\ n$, we are assured that

$$\int_a^b \log x dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \{\log\,(a + k\Delta x_k)\}\,\Delta x_k. \tag{3}$$

Let $a=r/s$, $b-a=p/q$, where p, q, r, and s are fixed positive integers. Let m be an arbitrary positive integer and define $n=mqs(b-a)=mps$. We note that p and s are fixed here, so that if $m\to\infty$, then $n\to\infty$. Now, $\Delta x_k=(b-a)/n=1/mqs$ and

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \{\log\,(a + k\Delta x_k)\}\,\Delta x_k \\ &= \frac{1}{mqs}\sum_{k=1}^{mps} \log\left(a+\frac{k}{mqs}\right) = \sum_{k=1}^{mps} \log\left(\frac{mqr+k}{mqs}\right)^{1/mqs}; \end{aligned} \tag{4}$$

$$e^{S_n} = \prod_{k=1}^{mps}\left(\frac{mqr+k}{mqs}\right)^{1/mqs} = \frac{1}{(mqs)^{p/q}}\left[\frac{(mqr+mps)!}{(mqr)!}\right]^{1/mqs}. \tag{5}$$

At this point, one introduces (proves) Stirling's formula, (2). Thus, if we define $Q(n)$ by the relation, $n!=Cn^{n+1/2}e^{-n}Q(n)$, then $\lim_{n\to\infty} Q(n)=1$. Using (2), we have

$$\frac{(mqr+mps)!}{(mqr)!} = \frac{Q(mqr+mps)(mqr+mps)^{mqr+mps}e^{-mps}}{Q(mqr)(mqr)^{mqr}}\left(1+\frac{ps}{qr}\right)^{1/2} \tag{6}$$

$$= \frac{Q(mqr + mps)(mqs)^{mps}e^{-mps}\left(\frac{r}{s} + \frac{p}{q}\right)^{mqr+mps}}{Q(mqr)(r/s)^{mqr}}\left(1 + \frac{ps}{qr}\right)^{1/2}.$$

Substitution of (6) into (5) yields

$$e^{S_n} = e^{-(p/q)}\left(\frac{r}{s} + \frac{p}{q}\right)^{(r/s)+(p/q)}\left(\frac{r}{s}\right)^{(-r/s)}F(p, q, r, s, m) \tag{7}$$

$$= e^{a-b}b^b a^{-a}F(p, q, r, s, m),$$

where

$$F(p, q, r, s, m) = \left[\frac{Q(mqr + mps)}{Q(mqr)}\right]^{1/mqs}\left(1 + \frac{ps}{qr}\right)^{1/(2mqs)}. \tag{8}$$

Let $m \to \infty$. Then $F(p, q, r, s, m) \to 1$, $n \to \infty$, and $S_n \to \int_a^b \log x dx$. Thus, from (7),

$$\int_{a=r/s}^{b=a+(p/q)} \log x dx = b \log b - a \log a - b + a. \tag{9}$$

Thus far, we have established the validity of (1) for *rational* numbers, $b > a > 0$. Because of the continuity property of the function, $\log x$, and because one may choose $a = \{r/s\}$ and $b = \{a + (p/q)\}$ as two rational sequences of numbers approaching two distinct, positive, irrational numbers as limits, respectively, the extension of the validity of (1) for *irrational* numbers is immediate. Thus, (1) is valid for all real numbers $b > a > 0$.

References

1. W. Feller, An Introduction to Probability Theory and Applications, New York, 1950, pp. 41–42.

2. M. I. Aissen, Some remarks on Stirling's formula, this MONTHLY, vol. 61, 1954, pp. 687–691.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

GUIDE ON PURCHASE OF EQUIPMENT

The Council of Chief State School Officers with the assistance of representatives from science, including staff members of the National Bureau of Standards, published in August, 1959 a very comprehensive guide for the purchase of equipment under the provisions of National Defense Education Act of

1958. The guide includes specifications on some 1000 items of equipment recommended for teaching science, mathematics, and modern foreign languages in elementary and secondary schools. There also is included an extensive bibliography for elementary and secondary school libraries. The project was in charge of Dr. Edgar Fuller, executive secretary of the Council. Advice on the equipment and bibliography recommended for use in the teaching of mathematics was given by a group of mathematicians and teachers whose meeting was made possible by the School Mathematics Study Group.

Copies of the publication may be purchased from the state superintendent of public instruction in each of the states, or directly from Ginn and Company.

AAAS STUDY ON THE USE OF SPECIAL TEACHERS OF SCIENCE AND MATHEMATICS IN GRADES 5 AND 6

One of the major activities in the extended Science Teaching Improvement Program of the American Association for the Advancement of Science, under the new three-year grant to AAAS by the Carnegie Corporation of New York, is to be known as the Study on the Use of Special Teachers of Science and Mathematics in Grades 5 and 6. The cooperation of the public school systems in Cedar Rapids, Iowa; Lansing, Michigan; Versailles, Kentucky; and Washington, D. C. have been obtained for this project. In each of these school systems, at least one special teacher of science and at least one special teacher of mathematics will teach only science or only mathematics in grades 5 and 6 to at least four different classes. The teachers have been selected by the school systems as those having special training, competence, and successful teaching experience in the areas to which they are assigned.

While the use of special teachers of art, music, and physical education is fairly widespread in American elementary schools, the question of special teachers in the academic disciplines such as science and mathematics remains highly controversial. Majority opinion of experts in elementary education, both administrators and teachers, seems to favor strongly the retention of the "self-contained" classroom at this level. On the other hand, many scientists favor the use of specially prepared teachers particularly because of the new and anticipated demands on the elementary teacher. It is quite impossible under the present methods of training elementary teachers for the elementary teacher to take what appears to be a necessary minimum of course work in science and mathematics, or in the other academic disciplines as well. The National Science Foundation sponsored curriculum studies, which are giving attention to the science curriculum at the elementary level, will stress how serious is the problem. Both the School Mathematics Study Group and the Biological Sciences Curriculum Study are concerned with grades from kindergarten to grade 8, as well as with the high schools, and there are also other privately sponsored activities which are working on materials at the elementary level.

An attempt will be made to provide a careful evaluation of the Study so

that interested persons will have available objective data to support their cases or to persuade them of weaknesses in their present point of view. It is anticipated that the Study will be conducted during a two-year period beginning in the fall of 1959 and it is to be hoped that the Study might be extended to more schools during the second year. (Abstracted from *Science Education News*, September 1959.)

A NEW KIND OF COURSE FOR THE PREPARATION OF TEACHERS*

E. P. Northrop, College of the University of Chicago

Three main streams can be discerned in current efforts to improve secondary school mathematics curricula in this country. First, various groups, working at both national and local levels, are formulating proposals for curricular change. Second, various groups, also working at both national and local levels, are preparing new classroom materials. Third, numerous programs, supported by industry and by private and public foundations, are being offered to improve the mathematical and professional competence of teachers.

These efforts have come into being and have grown too large too fast to be well coordinated. Yet their interrelatedness is readily apparent. If the proposals for curricular change are to be understood and not rejected out of hand, then teachers need to be trained in the relevant mathematical subject matter. If the proposals are to be acted upon and not just read and filed away, then teachers need to be supplied with relevant text materials.

These efforts, however well conceived and executed, will not in themselves suffice. What is lacking? The more thoughtful of the groups proposing curricular change agree that no single curriculum will serve all the students in all the schools. Accordingly, when they produce a particular curriculum by way of illustration of what might be done, they are quick to emphasize that this is only an example, and that the teachers of the country must adapt the suggested changes to their local needs. Again, the more thoughtful of the groups preparing new text materials similarly reject the notion of a universal panacea. Accordingly, when they produce texts by way of illustration of what might be written, they too are quick to emphasize that these are only examples, and that the teachers of the country must adapt the suggested materials to their local needs.

What is lacking now begins to emerge in the picture, as does the place where the lack can be remedied. Overburdened classroom teachers, however well trained or retrained, are about to be faced with a bewildering and often conflicting variety of curricular proposals and text materials. They need help in selecting and organizing materials for use in their own local classrooms. They need, in short, help in learning to think and to write for themselves. The help they need can and should be provided in the third stream of our current efforts, namely, in the programs of re-education being offered to inservice teachers.

* Presented at a panel discussion, "The Training of Secondary School Mathematics Teachers," Annual Meeting of the Mathematical Association of America, Philadelphia, Pa., January 23, 1959.

Let us therefore consider a possible design for a seminar of nine months' duration, intended to provide the individual teacher with a working understanding of curriculum construction, together with working experience in the preparation of materials for use by his own students. It is assumed that the seminar is to be attended by twenty-five or thirty teachers, and is to be conducted concurrently with, or subsequent to, appropriate courses in mathematics proper.

During the first three months of the year, and while the teacher is learning some of the mathematics he will need, the seminar would be devoted to a down-to-earth consideration of factors that have an important bearing on curriculum construction. Examination of a few specific curriculum proposals, including the bases offered in support of them, would lead naturally and quickly to a discussion of such factors as the needs or demands of society, the philosophy of education of the school, the characteristics of its students, the nature of the subject-matter, the formulation of aims or objectives, the selection and organization of learning activities, and testing procedures. Such a discussion, properly conducted, can serve three purposes. First, it can bring order into possibly disordered and unrelated ideas gleaned from earlier courses in education. Second, it can provide the teacher with a framework for the writing efforts to follow. Third, it can provide him with a rationale by which to justify his proposals and sell his product to an audience too often ignored, namely, the folks back home.

The next two months would be devoted to group writing efforts. The class would be divided into groups of four or five teachers each, sorted according to expressed interest in this, that, or the other general topic. Each group, having selected a modest and manageable unit, would settle upon the kind of student for which it is intended and what he is expected to achieve from it, would outline it, and would write a piece of text complete with problem material. It would be made clear at the outset that these initial efforts were not expected to terminate in a polished product—that they were simply practice for a more extensive effort to come. The seminar would continue to meet, though perhaps less frequently, for open discussion of progress, major difficulties encountered here or there, and so on. The final products would be reproduced and distributed to all members of the seminar for analysis and criticism, in the course of which all would learn something about the importance of correctness, clarity, and precision in mathematical exposition. But there is an additional and very real value in this preliminary writing. Here the teacher finds himself, perhaps for the first time, facing up to issues and committing himself to paper, and by doing so in the company of his fellows he gains as much in confidence as he gains in simple experience.

The last four months of the year would be devoted to the major effort and product of the seminar. Each teacher would embark on an individual project in the planning of a year course for use in his own high school by his own students. The project would issue in a substantial report consisting of the following parts:

1. A brief general description of the course.
2. Its justification, addressed to school board and principal, parents and fellow teachers, and including general statements of aims, of shortcomings of the present program, and of strengths of the program being proposed.
3. A full outline of the year's work.
4. A complete treatment of a substantial unit of the course (for example, a month's work, including classroom materials to be used by the student for further study), an examination over the unit, and a teachers' manual for use by other teachers.

During this period the seminar would be disbanded as such, though it could conceivably meet for discussions with visiting representatives of outside groups concerned with curricular proposals or preparation of text materials. Members of the staff would of course make themselves available for individual conferences. It would be desirable, if it could be arranged, to use the fourth and last month of the period for criticism and revision of the reports, especially if they were to be used—as well they might—in partial fulfillment of certain degree requirements.

A seminar of the kind described is perhaps most suitable for use with teachers who have had several years of teaching experience.† It could nevertheless be adapted—no doubt with lowered sights—for use with new teachers in the course of their practice teaching. Indeed, a foreshortened version of the seminar might be imagined for use in a summer program of, say, eight weeks' duration. Here the work would take the form of group projects, with a skilled leader assigned to each of the groups of four or five teachers for two purposes: to direct the writing efforts and, in the course of them, to call attention to the curricular issues involved. In this way the first two parts of the nine-month seminar described above might be successfully telescoped, especially if the several groups concerned were to meet together once a week, under a director of the entire operation, for discussion of current proposals for curricular change.

It is to be anticipated that the individual or group reports issuing from seminars of the kind described in this paper would be of mixed quality. But it can be safely asserted that the author or coauthor of even the weakest report would have grown enormously in critical understanding of his task. It is often said that one does not really know his subject until he tries to teach it. In the same vein, one does not really know his subject or his teaching until he tries to write what he teaches. Even if what he writes is not worth teaching, the experience will have made him a far better and wiser judge of courses and texts that others write for him to use.

† Evidence that it can be done is furnished by the fact that in the 1958–59 Academic Year Institute for High School Mathematics Teachers at the University of Chicago, Mr. Maurice Hartung and the author are in midstream with a second such seminar. The author's indebtedness to Mr. Hartung, and to other colleagues in education and mathematics who assisted in the design and conduct of the seminar, is hereby gratefully acknowledged.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1381. *Proposed by H. E. Hart, City College of New York*

Show that

$$1 + x + x^2/2! + x^3/3! + \cdots + x^{2n}/(2n)! = 0$$

has no real roots.

E 1382. *Proposed by Ian Connell, University of Manitoba*

Let u_n be the nth term of the sequence

$$1, 2, 4, 5, 7, 9, 10, 12, 14, 16, 17, \cdots,$$

where one odd number is followed by two evens, then three odds, etc. Prove that

$$u_n = 2n - [(1 + \sqrt{8n - 7})/2],$$

where square brackets denote the integral part.

E 1383. *Proposed by E. J. Burr, University of New England, N.S.W., Australia*

Let P_n be the problem: "Find a set of $n>2$ consecutive positive integers, the greatest of which is a divisor of the least common multiple of the remaining $n-1$." Show that precisely one of the problems P_n has a unique solution.

E 1384. *Proposed by J. H. Butchart, Arizona State College*

Construct a circle through two given points, separated by a given circle, which shall cut the given circle at the smallest possible angle.

E 1385. *Proposed by John Burke, Saranac Lake, N. Y.*

A homogeneous die is "loaded" at a single point and is cast n times. The one spot appears n_1 times, the two spot n_2 times, $\cdots$, the six spot n_6 times. Find the (probable) position of the load.

SOLUTIONS

Generalization of a Russian Olympiad Problem

E 1337 [1958, 708; 1959, 424]. *Proposed by the Student-Faculty Colloquium, Carleton College*

What is the largest value of n, in terms of m, for which the following sentence is true? If from among the first m natural numbers any n are selected, among the remaining $m-n$ at least one will be a divisor of another.

II. *Solution by J. L. Selfridge, IBM Research Center, Yorktown Heights, N. Y.* The published solution [1959, 424] contains an oversight. The proof is correct only for $m<12$. If $m\geqq 12$ and we select $n=[m/2]-1$ numbers, the remaining $m-n$ numbers need not contain 1 or $(k, 2k)$ as asserted by the solver. This is easily shown by selecting 1 and all numbers exactly divisible by an odd power of 2. For example, when $m=12$ select 1, 2, 6, 8, 10. The remaining numbers are odd or an odd multiple of 4, and no one is twice another. Following is believed to be a correct solution of the problem.

If $1, 2, \cdots, [m/2]$ are selected from $1, 2, \cdots, m$, then none of those remaining divides another of them. This shows that $n<[m/2]$. Now we show that for any $n\leqq[m/2]-1$, some a and b will remain unselected with $b=2^k a$. Suppose that no such a and b exist. Then we must select all but one from each sequence $r, 2r, \cdots, 2^s r$. These sequences are disjoint if r is odd, and we count those selected by noticing that selecting all but the largest in each sequence means selecting $1, 2, \cdots, [m/2]$.

Locus of Centers of a Family of Equilateral Triangles

E 1351 [1959, 141]. *Proposed by J. F. Darling, Woodstown, New Jersey*

Find the locus of the centers of the equilateral triangles whose sides a, b, c pass through three fixed points A, B, C respectively.

Solution by Helen M. Marston, Educational Testing Service, Princeton, N. J. Construct the three circles O_a, O_b, O_c for which the segments BC, CA, AB respectively subtend 120° arcs, the minor arc in each case being directed toward the inside of triangle ABC. From any point P_c on O_c draw P_cAP_b (line a) and P_cBP_a (line b). Angles P_cP_bC and P_cP_aC each equal 60° by construction, whence P_b, C, P_a are collinear (line c) and triangles $P_aP_bP_c$ represent one set of equilateral triangles abc. Let M_a, M_b, M_c be the midpoints of the minor arcs BC, CA, AB respectively. The center M of triangle $P_aP_bP_c$ is the intersection of P_aM_a and P_bM_b, which must intersect at 60°. Since the locus of M includes points M_a, M_b, M_c, these three points form an equilateral triangle and the locus of M is the circle circumscribing it.

If circles O_a', O_b', O_c' are constructed similarly but with minor arcs directed toward the outside of triangle ABC, we obtain a second set of equilateral triangles abc, a second set of minor-arc-midpoints M_a', M_b', M_c', and a circle circumscribing them for the locus of M'.

The required locus is therefore a pair of circles. It can be shown that they

have a common center G, which is the centroid of triangle ABC. In the limiting case where A, B, C are collinear, the two circles coincide. When A, B, C form an equilateral triangle, there is only one set of equilateral triangles abc and the locus of M is the single point G.

Also solved by Leon Bankoff, W. B. Carver, J. W. Clawson, Michael Goldberg, L. D. Goldstone, W. R. McEwen and Amos Nannini (jointly), D. C. B. Marsh, Beckham Martin, D. C. Stevens, and the proposer.

The loci of M and M' may be alternatively described as the circles having centers at the centroid of triangle ABC and passing through the isogonic centers R and R' of triangle ABC, or as the circumcircles of triangles $O_a'O_b'O_c'$ and $O_aO_bO_c$.

Goldberg, Marsh, and Martin used cartesian coordinates; Carver, Clawson, and the proposer used conjugate coordinates. Goldstone pointed out the connection between the problem and Problem 4301 [1950, 565]. Martin considered the more general problem of finding the locus of the centroids of all triangles of a given species whose sides a, b, c pass through three fixed points A, B, C respectively.

The Case of the Careless Typesetter

E 1352 [1959, 141]. *Proposed by C. W. Trigg, Los Angeles City College*

In setting the type for the multiplication $(abc)(bca)(cab) = 234235286$, $a > b > c$, in which the unit's digit is 6, the remaining digits of the product became pied. Restore them to their proper order.

Solution by W. R. Talbot, Lincoln University, Missouri. Let $n = (abc)$ and let N be the desired product. To establish limits on n, we notice that if $c = 1$, then the largest n, namely 981, gives $N = 159{,}080{,}922$, which is too small. Therefore $988 > n > 981$. But the sum of the digits of N is $35 \equiv 2 \bmod 3$; whence n and its permutations are congruent to 2 mod 3. Then n is 986 or 983, of which only 983 has a product of digits ending in 6. It is now readily found that

$$(983)(839)(398) = 328{,}245{,}326.$$

Also solved by A. N. Aheart, Merrill Barnebey, D. A. Breault, W. E. Buker, J. M. Calloway, G. B. Charlesworth, P. L. Chessin, R. J. Cormier, Eleanor G. Dawley, Monte Dernham, J. E. Faulkner, J. F. Foley, Michael Goldberg, L. D. Goldstone, H. W. Gould, Robert Gregorac, E. E. Griffee, P. G. Hodge, Jr., J. H. Hodges, J. M. Howell, A. R. Hyde, F. S. Innis, Jr., Sister Kenneth Kolmer, W. R. McEwen, D. C. B. Marsh, Helen M. Marston, J. H. Means, Franklin Mohr, D. L. Muench, J. B. Muskat, D. J. Persico, J. P. Phillips, J. L. Pietenpol, C. F. Pinzka, B. E. Rhoades, J. A. Rodriguez, J. L. Selfridge, D. C. Stevens, W. B. Stovall, Jr., Harry Weingarten, Mildred West, Charles Wexler, Dale Woods, Ronald Wyllys, and the proposer.

Eccentricity of an Elliptical Section of a Cylinder

E 1353 [1959, 141]. *Proposed by L. A. Kenna, University of Arizona*

What is the eccentricity of an ellipse formed by a plane cutting the axis of a right circular cylinder at an angle of θ?

Solution by A. N. Aheart, West Virginia State College. Denoting the lengths of the major and minor axes by $2a$ and $2b$, we have

$$e = (1 - b^2/a^2)^{1/2} = (1 - \sin^2\theta)^{1/2} = |\cos\theta|.$$

Also solved by R. G. Albert, E. F. Allen, J. W. Baldwin, Leon Bankoff, Merrill Barnebey, H. F. Bechtell, A. P. Boblétt, D. A. Breault, C. S. Carlson, G. B. Charlesworth, P. L. Chessin, A. G. Clark, R. J. Cormier, C. H. Cunkle, Alfred Diarnbey, Michael Difon, C. W. Dodge, Ernest Enns, G. W. Erwin, Jr., J. E. Faulkner, Susan L. Friedman, Todd Gitlin, George Glauberman, Michael Goldberg, L. D. Goldstone, F. D. Grogan, J. D. Haggard, R. Henry, Richard Holt, A. F. Horadam, R. H. Hou, A. R. Hyde, G. S. Innis, Jr., M. S. Itzkowitz, J. D. E. Konhauser, W. E. Lawrence, W. R. McEwen, D. C. B. Marsh, Beckham Martin, C. S. Ogilvy's freshmen calculus class, D. J. Persico, J. L. Pietenpol, C. F. Pinzka, B. E. Rhoades, L. A. Ringenberg, D. A. Robinson, J. A. Schumaker, Ruth G. Smith, D. C. Stevens, W. B. Stovall, Jr., W. R. Talbot, C. W. Trigg, T. C. Wales, Charles Wexler, L. H. Williams, R. H. Wilson, Jr., Alan Wofford, Dale Woods, J. W. Young, and the proposer.

Editorial Note. The eccentricity of the conic section formed by a plane cutting at an angle θ the axis of a right circular cone of generating angle ϕ is $|\cos\theta|/\cos\phi$.

A Property of the Coefficients of $(1+x)^{p-2}$

E 1354 [1959, 141]. *Proposed by P. L. Chessin, University of Maryland*

If $(1+x)^{p-2}=1+a_1x+a_2x^2+\cdots+a_{p-2}x^{p-2}$, where p is a prime, then a_1+2, a_2-3, a_3+4, $\cdots$ are all multiples of p.

Solution by C. F. Pinzka, University of Cincinnati. Multiplying corresponding members of the k congruences $p-i\equiv -i \pmod p$, $i=2, 3, \cdots, k+1$, gives

$$(p-2)!(p-k-2)!\equiv(-1)^k(k+1)! \pmod p.$$

For $k<p$ we have $(k!, p)=1$ and thus

$$(p-2)!/(p-k-2)!k!\equiv(-1)^k(k+1) \pmod p,$$

demonstrating that $a_k+(-1)^{k+1}(k+1)$ is divisible by p.

Also solved by A. N. Aheart, R. G. Albert, J. W. Baldwin, H. F. Bechtell, R. F. Brown and Anna Endelman and Joel Levy (jointly), J. M. Calloway, G. B. Charlesworth, A. E. Danese, Alfred Diarnbey, H. B. Emerson, J. E. Faulkner, N. J. Fine, Todd Gitlin, George Glauberman, Michael Goldberg, L. D. Goldstone, Joseph Hammer, Leonard Hauer, Richard Holt, J. H. Hodges, Irving Katz, Jack Klugerman, Sidney Kravitz, W. E. Lawrence, Robert McGuire, D. C. B. Marsh, Andrzej Mgkowski, Valerie Mikė, Mary P. Moseley, D. L. Muench, Stewart Nagler, K. K. Norton, C. S. Ogilvy, D. J. Persico, Benjamin Sapolsky, L. J. Schneider, Arnold Singer, D. C. Stevens, H. R. Stevens, Gerald Stoller, W. R. Talbot, W. A. Veech, T. C. Wales, Charles Wexler, Dale Woods, Ronald Wyllys, J. W. Young, David Zeitlin, and the proposer.

A Detail of a Larger Work

E 1355 [1959, 141]. *Proposed by A. J. Goldman, National Bureau of Standards*

If $y(x)>0$ and $y=\log[1+(\log x)/x+y/x]$ for all sufficiently large x, prove that $y\sim(\log x)/x$ as $x\to\infty$. (This is a step whose details are left to the reader in a paper by S. Chowla and F. C. Auluck, *Some properties of a function considered by Ramanujan*, J. Indian Math. Soc., vol. 4, 1940, pp. 169–173.)

Solution by David Zeitlin, Remington Rand UNIVAC, St. Paul, Minn. From

$$1+(\log x)/x+y/x=e^y>1+y$$

we have

$$0 < y < (\log x)/(x - 1),$$

whence $y(x) \to 0$ as $x \to \infty$. Therefore

$$\lim_{x \to \infty} xy/(\log x) = \lim_{y \to \infty} y/(e^y - 1) = 1.$$

Also solved by D. A. Breault, R. F. Brown and Anna Endelman and Joel Levy (jointly), N. J. Fine, George Glauberman, Walter James, D. C. B. Marsh, Morris Morduchow, K. K. Norton, C. F. Pinzka, D. C. Stevens, Peter Treuenfels, and W. A. Veech.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4863. *Proposed by W. F. Stinespring, Institute for Advanced Study*

For what constant k (if any) can the inequality

$$|A + B| \leqq k(|A| + |B|)$$

be established for all n by n real symmetric matrices ($n \geqq 2$)? Here the notation $|A|$ means the positive square root of A^2, and the relation $\leqq$ between two matrices means that their difference is nonpositive definite.

4864. *Proposed by D. J. Newman, Massachusetts Institute of Technology and AVCO Research and Development*

Let a coin be tossed repeatedly, and let p be the probability that at some point the number of heads exceeds twice the number of tails. Prove

$$p = \tfrac{1}{2}(\sqrt{5} - 1).$$

4865. *Proposed by L. Lewin, Enfield, England*

Defining the inverse tangent integral of the second order by

$$Ti_2(x) = \int_0^x \frac{\tan^{-1}(x)}{x}\,dx = \frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} \cdots,$$

prove that $6Ti_2(1) - 4Ti_2(1/2) - 2Ti_2(1/3) - Ti_2(3/4) = \pi \log 2$.

4866. *Proposed by F. D. Parker, University of Alaska*

Show that a semi-magic matrix A (the sums of the rows and columns are all equal) can be decomposed into a sum $B+C$ such that for integral K,

$$(B + C)^K = B^K + C^K.$$

4867. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let f be a measurable function such that $0 < f(x) \leqq 1/2$ on $[0, 1]$. Prove

$$\frac{\exp \int_0^1 \log f(x)dx}{\int_0^1 f(x)dx} \leqq \frac{\exp \int_0^1 \log (1 - f(x))dx}{\int_0^1 (1 - f(x))dx}.$$

4868. *Proposed by Ronald Pyke, Columbia University*

Show that for all $\beta \leqq e^{-1}$ and for all real x the function f defined by $f(0, \beta) = 1$ and

$$f(x, \beta) = x \sum_{j=0}^{\infty} \beta^j \frac{(j + x)^{j-1}}{j!} \qquad (x \neq 0)$$

is finite and satisfies the relationship

$$f(x, \beta) = e^{x\beta f(1,\beta)}.$$

SOLUTIONS

Matrices Having a Given Multiplication Table

3099 [1924, 455]. *Proposed by S. A. Corey.*

Let H, I, J, and K be the complex quaternion units,

$$\tfrac{1}{2}\{1 - j - \theta(i + k)\}, \quad \tfrac{1}{2}\{1 + j + \theta(i - k)\},$$
$$\tfrac{1}{2}\{1 + j - \theta(i - k)\}, \quad \tfrac{1}{2}\{1 - j + \theta(i + k)\},$$

respectively, which have the multiplication table

	H	I	J	K
H	H	I	H	$-I$
I	H	I	$-H$	I
J	J	$-K$	J	K
K	$-J$	K	J	K.

Prove that there exists a set of four matrices, involving no imaginaries, which have the same multiplication table.

Solution by D. C. B. Marsh, Colorado School of Mines. Using known correspondences of $a+bi+cj+dk$ and $x+\theta y$ to

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

respectively, we can establish composite 8 by 8 matrices with real components which have the required multiplication table. One set follows:

$$H = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \qquad I = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$J = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \qquad K = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Linear Independence

4797 [1958, 452; 1959, 322]. *Proposed by D. J. Newman, Massachusetts Institute of Technology and AVCO Research and Development*

Prove that all expressions like $7\sqrt{19}/4-3\sqrt{7}+8\sqrt{6}/5$ are irrational. More specifically, prove that the square roots of the square-free integers are linearly independent over the rationals.

II. *Editorial Note.* P. T. Bateman has raised the objection to the solution printed in the April issue that the c_i appearing in formula (3) may be all zero and that a proof to the contrary is not given. A similar oversight seems to exist in each of the other contributions to this problem, so that we have no solution available at this time.

Bateman also notes that the result of the problem is contained in Theorem 2 and its Corollary in A. S. Besicovitch, *Linear independence of fractional powers of integers*, Proceedings of the London Mathematical Society, XV (1940), pp. 3–6.

Reducibility of a Polynomial

4815 [1958, 713]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $a_1, \cdots, a_n$ be distinct integers. Discuss the reducibility of the polynomial $f(x)=(x-a_1)\cdots(x-a_n)+1$ over the field of rationals.

Remarks by Hans Rademacher, University of Pennsylvania. This problem is 50 years old and due to I. Schur. A solution is printed in Pólya-Szegö, *Aufgaben und Lehrsätze*, vol. 2, VIII, pp. 346–7. Many generalizations have been discussed in the meantime. References can be found in Brauer and Ehrlich, *On the irreducibility of certain polynomials*, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 844–856.

Also solved by P. T. Bateman, J. H. Hodges, D. C. B. Marsh, W. F. Trench, and the proposer. The result is: the polynomial is irreducible except when $n=2$ and a_1 and a_2 differ by 2, and when $n=4$ and a_1, a_2, a_3, a_4 are in arithmetic progression with common difference 1.

An Integral Equation

4816 [1958, 779]. *Proposed by M. S. Klamkin, AVCO Research and Development, Wilmington, Mass.*

Solve the integral equation

$$\int_0^\infty t^3\phi(x-t)dt = a\left\{\int_0^\infty t^2\phi(x-t)dt\right\}^b,$$

where a and b are independent of x.

Solution by Robert Weinstock, University of Notre Dame. Introducing the change of variable $u=t-x$ and defining F by

$$F(x) = \int_0^\infty t^2\phi(x-t)dt = \int_{-x}^\infty (u+x)^2\phi(-u)du, \tag{1}$$

we may differentiate the integral equation and obtain

$$3\int_{-x}^\infty (u+x)^2\phi(-u)du = ab\left\{\int_{-x}^\infty (u+x)^2\phi(-u)du\right\}^{b-1}\cdot 2\int_{-x}^\infty (u+x)\phi(-u)du,$$

that is

$$3 = ab[F(x)]^{b-2}F'(x) = ab(d/dx)\{(b-1)^{-1}[F(x)]^{b-1}\},$$

provided $b\neq 1$; the case $b=1$ is handled separately below. We introduce $m=b/(1-b)$ and integrate to obtain

$$F(x) = [c-(3/am)x]^{-m-1}, \tag{2}$$

where c is an arbitrary constant.

Differentiating (1) three times and using (2) we obtain

$$\phi(x) = \tfrac{1}{2}F'''(x) = \tfrac{1}{2}(3/am)^3(m+1)(m+2)(m+3)[c-(3/am)x]^{-m-4}.$$

For the existence of the integrals involved we must have $m>0$ (whence $0<b<1$), $a>0$, and $x<amc/3$.

In case $b=1$, the differential equation for F reads $F'(x)=(3/a)F(x)$, whence $F(x)=2(a/3)^3Be^{3x/a}$, where B is an arbitrary constant. We then have $\phi(x)=\frac{1}{2}F'''(x)=Be^{3x/a}$, which satisfies the integral equation for $b=1$, provided $a>0$, for all x.

If $a=0$, there is clearly only the trivial solution $\phi(x)=0$.

Also solved by H. E. Fettis, Emil Grosswald, C. C. Yalavigi and the proposer.

Entire Functions

4817 [1958, 779]. *Proposed by I. S. Gâl and Stanley Kaplan, Cornell University*

Determine all entire functions $f(z)$ such that $|f(z)|=1$ whenever $|z|=1$.

Solution by C. Goffman and M. Henriksen, Purdue University. Write $f(z)=z^ng(z)$, where n is a nonnegative integer, g is entire, and $g(0)\neq 0$. Since $|z|=1$ implies $|g(z)|=1$, either g is constant, or the open mapping g sends the unit circle onto itself. In the latter case, the Schwarz reflection principle tells us that if z_1 and z_2 are collinear with the origin and $|z_1z_2|=1$, then

$$|g(z_1)g(z_2)| = 1.$$

(*cf.*, *e.g.*, Bieberbach, *Conformal Mapping*, New York, 1953, p. 76). It follows that g is bounded on a neighborhood of infinity. So, by Liouville's theorem, g is a constant.

Hence $f(z)=az^n$, where $|a|=1$.

Also solved by I. N. Baker, P. T. Bateman, Harley Flanders, D. S. Greenstein, Benoit Lachapelle, W. F. Trench, and the proposers.

A Property of the Miquel Circle of a Complete Quadrilateral

4818 [1958, 779]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let d_i be the sides of a complete quadrilateral, and A_{ij} be the vertex on d_i, d_j. Let t_i be the triangle formed by the sides other than d_i, and (O_i) denote the circumcircle of t_i. Denote the Simson line of a point S_i of (O_i) with respect to t_i by D_i.

Then prove that, if D_i and d_i are parallel for all i, (1) the line S_iO_p passes through the vertex A_{qr} $(i,\ p\neq q,\ r)$, and (2) the points S_i all lie on the Miquel circle (O).

Solution by the proposer. (1) Let the projections of S_i and O_p on d_r be denoted by U_{ir}, V_{pr}. Then, using directed angles, we have

$$\begin{aligned}
\sphericalangle V_{pr}O_pA_{qr} &= \sphericalangle\, A_{ir}A_{qi}A_{qr} \quad \text{(from the circle } (O_p)\text{)},\\
&= \sphericalangle\, U_{ir}U_{iq}A_{qr} \quad \text{(since } D_i \text{ is parallel to } d_i\text{)},\\
&= \sphericalangle\, U_{ir}S_iA_{qr}. \quad \text{(from the circle } S_iU_{ir}A_{qr}U_{iq}\text{)}.
\end{aligned}$$

Now, since O_pV_{pr} is parallel to S_iU_{ir}, we get the required collinearity of S_i, O_p, A_{qr}.

(2) Since the line of centers O_pO_q is perpendicular to the radical axis FA_{pq}, where F is the Miquel point, we have successively

$$\begin{aligned}
\sphericalangle O_pO_rO_q &= \sphericalangle A_{qi}FA_{pi}, \\
&= \sphericalangle A_{qr}A_{pq}A_{pi} \qquad \text{(from the circle } (O_r)\text{)}, \\
&= \sphericalangle A_{qr}A_{pq}A_{pr}, \\
&= \sphericalangle A_{qr}S_iA_{pr} \qquad \text{(from the circle } (O_i)\text{)}, \\
&= \sphericalangle O_pS_iO_q, \qquad \text{(from property (1))},
\end{aligned}$$

and S_i lies on the Miquel circle (O).

Also solved by A. E. Landry.

Sum of Five Distinct Squares

4819 [1958, 779]. *Proposed by P. T. Bateman, University of Illinois*

Suppose k is a given nonnegative integer. Show that every sufficient large positive integer is a sum of five squares none of which is less than k^2 and no two of which are equal.

Editorial Note. After having submitted an independent proof, the proposer finds stronger results obtained by E. M. Wright, using Vinogradov's method. [Quart. J. Math. Oxford Ser. (2), vol. 4, 1933, pp. 37–51.] Wright showed that every large positive integer can be expressed as a sum of five squares whose ratios are roughly prescribed in advance. This certainly implies the present theorem.

Also solved by L. Carlitz, using a result of Walfisz'.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Mathematics in Business. By Lloyd L. Lowenstein. Wiley, New York, 1958. xv+364 pp., $4.95.

"This text is for use in a first course in the mathematics of business. The intention is to give the student a firm foundation for further courses in the mathematics of finance as well as accounting, business statistics, insurance, and other business subjects." The book should be successful in accomplishing the stated aims of the author. Professor Lowenstein has a gift of writing in clear and simple, as well as concise, English. Apparently the students for whom this book

was written are those without exceptionally strong backgrounds in high school mathematics, and although the degree of rigor attained is superior to that in many similar books, the approach is not too subtle for his audience.

Chapters I and II review arithmetic and beginning high school algebra. This material is not an integral part of a course for those with normal preparation. Logarithms and the Binomial Theorem are treated in appendices whereas progressions are taken up when needed in the body of the text. The business mathematics chapters cover the topics: percentage; profit and loss; simple interest and discount; compound interest; depreciation; annuities with applications to amortizations, sinking funds, and bonds. The book is not intended to be a treatise on the theory of interest and Professor Lowenstein has achieved a splendid balance between theory and practice. More attention is given to simple interest methods than in some books.

Unusual features of the book are the listing of the answers to the odd-numbered exercises at the end of each chapter and very effective use of footnotes for clarification of theory. The text includes 61 pages of the usual financial tables. The logarithm table is only a five-place table but the logarithms of interest rates are given to seven decimal places.

C. L. SEEBECK, JR.
University of Alabama

Experimental Designs in Industry. Edited by Victor Chew. Wiley, New York, 1958. 268 pp. $6.00.

This is a collection of some of the papers given at a symposium held November 5–9, 1956, conducted by the Institute of Statistics of the University of North Carolina, State College Section, and sponsored by the Mathematics Division of the Air Force Office of Scientific Research.

The papers are of a technical, statistical nature but use relatively elementary mathematics. Up-to-dateness is the keynote. Parallel treatments of experimental design in general, factorial designs, and designs for exploring response surfaces are given via theoretical discussions, examples of applications, and extensive bibliographies. Ample detail is given so that the reader can understand the differences between experimental problems in industry and agriculture (the origin of many of the designs and the vocabulary) and so that the special techniques discussed here can be used by the working statistician.

I. RICHARD SAVAGE
University of Minnesota

A Modern Approach to Intermediate Algebra. By Henry A. Patin. Putnam's, New York, 1958. 224 pp. $3.75.

According to the preface, this text is a one-semester modern approach to intermediate algebra, with two major purposes. The first is "to acquaint the student with a logical system as it appears in mathematics." The preface also states that the topics and their order of presentation are fairly traditional and

that the presentation, although not rigorous, is coherent.

Thus, the major new contribution which this book purports to make is the development of algebra as a logical system. However, most of this is contained on pages 6 through 19, where the postulates for the integers, then the postulates for the rationals are stated and the basic rules of ordinary algebra are "proved." The treatment is very terse and abstract and is begun without warning, motivation, or any attempt to relate the laws to the student's past experience. The only hint that something is afoot is the assertion, "We will now state our fundamental assumptions." The next thing the reader sees is

"1.1 $a+b=b+a$"

Unlike certain recent texts on the collegiate level, the problems make no contribution toward preparing the student for the next abstract formulation, or in helpng him actually devise it; they are mere practice in identifying the laws. Here is one: "Show what laws have been used in $-3-2=(-3)+(-2)=-1(3+2)$." While a person acquainted with modern abstract algebra might like this presentation in chapter 1, the novice would probably find Chapters 1 and 2 of Birkhoff and MacLane more accessible.

Chapter 1 contains the author's major effort at infusing modern ideas into his work. Thereafter, the book grows more traditional, chapter by chapter. By the time the one on coordinate geometry was written, modern ideas were abandoned completely. For example, the introduction of the concept of function is shockingly inadequate for a text published in AD 1958: "An algebraic expression in $x-3/2x+7/2$ is called an algebraic function of x." There is no definition beyond this.

There are few imaginative or challenging problems. Most could have been copied directly from the easier parts of traditional texts. Since one of this book's claims to the modern spirit is that excessive manipulation is avoided, the lists of problems contain only those involving easy manipulations. Consequently the book would not satisfy the teacher of a traditional course, who would consider it watered down.

The final chapter is a four and one-half page discussion of algebra as a deductive system. This might be clear enough for a mature reader, but is certainly impenetrable to most 11th-grade high school students.

All prospective authors of modern texts should read this work, since it indicates how much thought must be given to a pioneering work, by its many evidences of having been poorly thought out: little or no motivation, off-the-cuff lists of problems, language unsuited to the level of 11th-grade students, pretentiously abstruse proofs, introduction of concepts such as *variable* and *function* without actual definitions, etc.

Another point strongly made by this book is that the writing of a modern text should *not* be undertaken unless the author has made contemporary think-

ing in all branches of elementary mathematics a part of his working knowledge.

One final comment. This book could be used in the second semester of a traditional 9th-grade course.

The type, paper and layout are superior; for this the publishers should be commended.

ED WALTERS
William Penn High School
York, Pennsylvania

Calculus. By Walter Leighton. College Mathematics Series. Allyn and Bacon, Boston, 1958. x+373 pages. $6.95.

The publication of this new volume in the College Mathematics Series follows by approximately a year that of the textbook *Calculus with Analytic Geometry* by Johnson and Kiokemeister in the same series. The Johnson and Kiokemeister text pleased many instructors, but had the disadvantage of being too difficult for most students. It contained perhaps an excessive emphasis on rigor. The present text goes too far in the other direction, and the author omits proofs which he should include. No proofs are given for the standard limit theorems, or for the existence of the definite integral. The proof that $\lim_{x\to 0} (\sin x)/x = 1$ is based upon the assumption that $\cos x$ is continuous at $x = 0$, and the author does not justify this assumption. The proof given for the Chain Rule is the one which most modern instructors try to avoid, and the lack of generality of this proof is pointed out by the author in a Critique. The author does not empashize that the derivative of a function $f(x)$ is taken with respect to x. Although the definition of a derivative is given on page 14, the idea that we differentiate $f(x)$ with respect to the variable x is apparently not mentioned until page 58, after the discussion and proof of the Chain Rule and the subsequent introduction of the differential. In short, the able student will feel that there is material which he needs, but which the author has not given him. His discouragement will be increased when he consults the references listed at the end of Chapter 1, and finds them far beyond him. An excellent reference text for this course is the Johnson and Kiokemeister book referred to above.

In view of the number of proofs which are omitted, it seems unnecessary to include an outline of "the basic algebraic theory of the representation of a rational function in terms of partial fractions" (see pp. 183–186).

Much of the discussion is too scanty. The chapter on Solid Analytical Geometry contains no adequate discussion of any surfaces except the plane, or any curves except the straight line, and contains not a single problem on quadric surfaces. The discussion of partial derivatives of functions of functions is rushed, and the reader is given the impression that if he wishes to learn the subject adequately he should consult other textbooks. Infinite series are motivated by Taylor's formula with the remainder. No discussion is given of operations with series. The book contains a short chapter on differential equations.

The writing style is good, and the book will be easy for most students to read. The problems appear in general to be well chosen. Problem 7, page 38 is incorrectly stated, while most students will abandon Problem 5, page 50 before they see the point to it.

DICK WICK HALL
Harpur College,
State University of New York

A First Course in Statistics. By R. Loveday. Cambridge University Press, London, 1958. x+121 pp. $1.75.

This is a nonmathematical introduction to descriptive statistics. It does not go very far nor does it give a very full discussion of any of the topics it contains. It does have worked-out illustrations that might be useful for an untrained person operating a desk calculator. It would not be useful in a course in statistics having any mathematical prerequisite. In the words of the author, "It may . . . prove helpful to students . . . who require an elementary introduction to the subject before embarking upon a more mathematical treatment."

WILLIAM G. MADOW
Stanford Research Institute

Some Aspects of Multivariate Analysis. By S. N. Roy. Wiley, New York, 1957. viii+214 pp. $8.00.

In the words of the author, this monograph does not attempt to cover the entire area of multivariate analysis but, rather, to deal with some important developments with which he has been associated during recent years. In particular, he omits a general discussion of such topics as factor analysis, classification problems and the multivariate counterpart of analysis of variance. Some of this omitted material will be treated in a future monograph.

The first two-thirds of the book deals mainly with obtaining confidence bounds for certain parametric functions. The testing of hypotheses in so far as it is developed is mainly a means to this end. The parametric functions under consideration are a set of natural measures of departure from the usual null hypothesis, there being a single such function in some cases and a set of such functions in more complicated cases.

The last third of the book consists of appendixes concerning matric theory, quadratic forms, linear transformations, jacobians and integration theory. The last two pages contain fifty-five references.

The author is to be commended for the excellence of this monograph. The printing is also very good. It is somewhat regrettable however that it is published on $8\frac{1}{2}'' \times 11''$ paper rather than a more standard size.

PAUL M. HUMMEL
University of Alabama

Scientific Programming in Business and Industry. By Andrew Vazsonyi. Wiley, New York, 1958. xix+474 pp. $13.50.

In this book the author states in the preface "what he (the reader) *can* expect to get is a thorough understanding of what scientific programming is and what it means to the business world." The book is addressed to the business man and the writer claims success in presenting the material to business men and confidence that this book, seriously studied, is suitable for business men. While the reviewer agrees that there are some business men who may read this book he feels that it is by far a better book for a college graduate well grounded in statistical theory and mathematics than it is for the average business man of his acquaintance.

The book contains chapters on mathematical models, three on linear programming, one each on convex programming, dynamic programming and game theory. Several chapters on various kinds of applications are also included. Among these are statistical inventory control, transportation allocation, and three chapters on scheduling. The treatment is expository and the author goes to some length to point out problems not solved by the devices at hand. While matrices are avoided generally the author does not explain notations for sums nor his nonstandard use of $x_{1,1}$ for x_{11} for example. Hence the book will not be simple for anyone not familiar with certain concepts of linear equations and finite dimensional linear spaces. The author is to be commended for the care with which he states assumptions.

The book should be valuable for college and high school teachers of mathematics who need to have some inkling of the applications of mathematics. It should be pointed out that the kind of applications given have been carried out mainly by industrial giants and not by small business men. Scientific programming is still in its infancy as far as both theory and practice go.

PRESTON C. HAMMER
University of Wisconsin

Applied Mathematics for Engineers and Physicists. (2nd ed.) By Louis A. Pipes. McGraw-Hill, New York, 1958. xi+723 pp. $9.50.

This well-written text is for the student of engineering or physics. It contains an amazing number of topics in a clear, concise style, thus making it a possible introductory source book for mathematical methods applicable to the engineering sciences. It does not treat topics depending on probability. It leaves the proof of basic mathematical theorems and the elaboration of special cases to mathematical analysis and advanced treatments. Much of it is at a slightly higher level than that of the student just out of a meager calculus course. There is considerable new material not found in the first edition.

The first chapter covers infinite series up through uniform convergence on a slightly higher level than the beginning calculus. Theorems are well stated. Dis-

cussion and applications bring out clearly their significance. The use of operators on numerous occasions is started here with the exponential operator form for Taylor's series. Each chapter closes with exercises and references.

A short chapter giving the elementary functions of complex numbers prepares for the development in the following chapter of Fourier series and the Fourier integral in complex form.

The fourth chapter treats linear algebraic equations, determinants, and matrices. The elementary properties of determinants are covered very rapidly. A numerical method due to F. Chio for evaluating large determinants is given. The work on matrices is far more complete than expected. Silvester's theorem permits the elegant solution of a system of linear differential equations in terms of the exponential function of a matrix. The confluent cases, where this method becomes especially valuable, is left to the literature.

Following a chapter on Newton's and Graeffe's method for solving equations comes one on linear differential equations with constant coefficients. Chapters 7, 8, and 9 are on "Oscillations of Linear Lumped Electrical Circuits," "Vibrations of Elastic Systems with a Finite Number of Degrees of Freedom," and "The Differential Equations of the Theory of Structures." Here is a profusion of applications of the earlier methods and some extensions. For example, one finds the analysis of a general network in matrix notation using Laplace transforms, Lagrange's equations, stability conditions, and Rayleigh's method for calculating frequencies.

For systems having repeated components and for numerical work there is a treatment of finite differences and difference equations. The work on partial differentiation has a discussion of maxima and minima, including Lagrange multipliers, and the evaluation of some definite integrals. Material on gamma, beta, and error functions prepare for Bessel functions and Legrendre Polynomials. Legendre functions of the second kind are omitted.

In "Vector Analysis" there is a clear discussion of curvilinear coordinates. Gauss's theorem and Stokes's theorem are used to find divergence and curl in curvilinear systems. The fundamental equations of hydrodynamics and of heat flow in solids are developed. The chapter closes with an introduction to the basic tensors and their applications.

The next three chapters cover the wave equation, Laplace's equation, and the equation of heat conduction.

The first chapter on complex variables goes through Laurent expansions, use of residues, and the argument principle. The other carries mapping through Schwartz's transformation and the applications of conjugate functions to potential and hydrodynamic problems.

Chapter 21 gives a more extended treatment of Laplace transforms. The p-multiplied form is used so as "to retain the operational forms of Heavyside." A heuristic treatment of the Fourier-Mellin theorem permits finding inverse transforms by residues. There are applications to the impulsive functions, partial differential equations, and integral equations. An appendix gives the basic

working theorems and a ten-page table of the transforms "most frequently occurring in the transient solution of electrical problems."

Some methods of treating nonlinear oscillatory problems are given in the last chapter. Some of the topics are "Forced Vibrations of Nonlinear Systems," "The Method of Kryloff and Bogolieuboff," and "Relaxation Oscillations." This 61-page chapter closes with 20 references, the most recent having a 1956 date.

It is a very useful book.

EARL LAFON
University of Oklahoma

Polynomial Expansions of Analytic Functions. By R. P. Boas, Jr. and R. C Buck. Springer-Verlag, Berlin, 1958. viii+77 pp. 19.80 DM.

This little book differs from most items in the Ergebnisse series in that, rather than being a survey of a field, it is a semi-expository presentation of the authors' approach to a topic.

By a basic set $\{p_n\}$ of polynomials is meant a Hamel basis of the vector space of all polynomials. This book deals with classes of analytic functions f which have expansions of the form $f(z) = \sum a_n p_n(z)$, where the sequence $\{a_n\}$ is defined in a prescribed manner. One such procedure, exploited by Whittaker and others, consists simply of substitution in the power series for f, so that $a_n = \sum \pi_{kn} f^{(k)}(0)/k!$, where $z^k = \sum \pi_{kn} p_n(z)$, the last summations all being finite since $\{p_n\}$ is a basic set. The authors' point of view allows other possibilities for the coefficients. The main idea is that of kernel expansion, which we describe briefly.

Given $\{p_n\}$, let $\{u_n\}$ be a sequence of functions, and define the kernels $K(z, w) = \sum p_n(z) u_n(w)$. Let $T: F \to f$ be a function-to-function-transformation defined by $f(z) = 1/(2\pi i) \int_\Gamma K(z, w) F(w) dw$, Γ being a suitably chosen path. Then $f(z) = \sum a_n p_n(z)$, where $a_n = 1/(2\pi i) \int_\Gamma u_n(w) F(w) dw$. The simplest example is that for which $p_n = z^n$ itself. Two classical choices for u_n are $u_n = w^n$ and $u_n = w^n/n!$, so that the respective kernels are $(1-zw)^{-1}$ and e^{zw}. With proper choice of F, these kernels yield the Taylor series and the Pólya representation.

The method is applied to a wide class of basic sets which the authors call generalized Appell polynomials. Many of the polynomial sets in the literature are special cases. Particular attention is given to Appell, Sheffer, and Brenke-Huff polynomials. Expansion theorems are obtained involving Borel and Mittag-Leffler summability as well as convergence. The form of the typical theorem is to characterize the class of functions for which the particular type of polynomial expansion converges to the function in the specified sense.

While the book is by no means light reading, it should be within the scope of a large segment of the mathematical public. In particular, it should make an excellent seminar topic.

CASPER GOFFMAN
Purdue University

Introductory Mathematical Analysis. By Edgar D. Eaves and Robert L. Wilson. Allyn and Bacon, Boston, 1958. xv+424 pp. $5.95.

This book is intended to serve the needs of freshmen enrolled in a terminal course meeting three hours a week for one academic year. The students are expected to have completed at least one year of high school mathematics; however, because so much is usually forgotten before the students begin their college mathematics, the book opens with a chapter of 37 pages providing a rather thorough review of the fundamental operations of arithmetic and algebra. The following seven chapters are devoted to an exposition of the concepts of function, instantaneous rates, differentiation and integration, with additional topics in algebra introduced when they are needed. Then follow chapters on trigonometry, simple and compound interest, the exponential and logarithmic functions and their derivatives, sequences, progressions and annuities, and a brief chapter on probability.

The text is well written in a style that is both lucid and thorough. The authors amply demonstrate their teaching skill by anticipating students' difficulties and by the care with which definitions and theorems are formulated. In the reviewer's estimation the authors can be commended for their good judgment in the selection of applications likely to interest students at this level and for achieving a standard of rigor considerably higher than in most textbooks of this category. The exercises, however, are nearly all quite simple, adequate for the run-of-the-mill student, but not sufficiently challenging to the abler ones.

J. M. FELD
Queens College

Calculus for Electronics. By A. E. Richmond. McGraw-Hill, New York, 1958. viii+407 pp. $6.00.

This book is designed for use as a mathematics text in technical institutes and training-in-industry programs. As the title indicates, it is primarily concerned with applications. Very few of the fundamental concepts of the calculus are described in terms sufficiently precise to be understandable. Many of its mathematical assertions are too vague to be called statements; a goodly number are false if taken literally. In summary, it is not a mathematics book.

If it is possible at all for a student to learn the basic techniques of the calculus from this book, he can do so only by working his way through the large number of illustrative examples and exercises provided by the author. His fear of the calculus may be allayed by the fact that most of these examples are selected from the subject matter of electronics. Can a student learn the calculus well enough to apply it only by learning how to do type-problems? This reviewer doubts it.

M. HENRIKSEN
Purdue University

Studies in the Mathematical Theory of Inventory and Production. By Kenneth J. Arrow, Samuel Karlin, Herbert Scarf. Stanford University Press, Stanford, 1958. x+340 pp. $8.75.

This book is a collection of papers exceptionally well inter-related. In part I, after giving a historical background, the nature and structure of the inventory problem are discussed. Part I ends with extensive summaries of the remaining fourteen chapters. Part II considers the finding of optimal policies in deterministic inventory processes. Part III discusses the same for stochastic inventory processes. Part IV concerns operating characteristics of inventory policies.

The mathematical tools used range from statistics through the calculus of variations, including differential equations and programming.

As the authors state in Chapter 2, they are interested only in existence theorems, etc. as long as they "are useful tools toward arriving at effective solutions," that is "effectively computable solutions."

This book is a welcome addition to the literature of applications of mathematics in business.

ALBERT NEWHOUSE
University of Houston

An Introduction to the Theory of Integration. By A. C. Zaanen. Interscience, New York, 1958. ix+254 pp. $7.25.

The author is to be commended for succeeding in the difficult task of writing a good book, of readable length, on modern integration. One of its most welcome and noteworthy features, which adds greatly to its value and usefulness, is its inclusion of certain related topics from classical and functional analysis, in which the theory has applications.

In developing the subject, the author tends to strike a middle position between the completely measure-theoretic approach and the more abstract treatment in which the integral is introduced as a linear functional. He treats the basic construction and extension problem in the following fashion. The existence of "elementary" integrals of the Daniell type is established by the methods of measure theory. The problem of extending an arbitrary elementary integral, defined on a linear lattice of real valued functions, is then reduced by the use of ordinate sets, to the problem of extending measures on semi-rings to the σ-rings they generate; and this latter problem is solved by the classical technique of outer measures. As a matter of personal taste, this reviewer would have preferred a treatment independent of ordinate sets.

After proving the principal properties of the extended integral and discussing the various relations between integrals derived abstractly from linear functionals and those arising more concretely from measures, the author proves Fubini's theorem in the form given by Stone. He then gives some applications of the theorem by establishing some results on integration by parts, the gamma function, and fractional integration.

Numerous other topics from analysis and functional analysis, including chapters on ergodic theory, unitary transformations on L_2, the Radon-Nikodym theorem, and Lebesgue's theorem on the differentiation of an indefinite integral, to mention a few, are also in the book. In addition, the book contains an excellent collection of exercises.

R. A. KUNZE
Massachusetts Institute of Technology

Zahlwort und Ziffer, Bd. II. By Karl Menninger. Vandenhoeck and Ruprecht, Gottingen, 1958. 314 pp. $6.00 (D.M. 24.80) Paperbound.

This is the second volume comprising the second part of the author's original treatise, *A Cultural History of Number*. The material of this book can be found scattered in other books and journals. The present treatment is a thorough, scholarly, and very readable coherent logical-historical development. The history of the development of number symbols and computation, both as things and as written symbolic manipulations, becomes an exciting drama, which is made all the more vidid by hundreds of diagrams, photographs, and drawings.

The treatise begins with the use of fingers, both to represent numbers up to 100,000, and to do simple operations on numbers. This is followed by a chapter on the way common people represented number by sticks, ropes, and other material things, leading to the use of letter symbols and finally unique numerical symbols. The way in which abaci and counting boards paved the way to the various algorisms performed with written symbols completes the story for the Western world. A final chapter treats the development of number language and number writing in China and Japan.

Although written in German, it is simple, yet elegant in style, and can be read easily by a second-year student of the language. Volumes I and II, bound together, are now available for $9.50 (D.M. 40.00).

HOWARD F. FEHR
Teachers College, Columbia University

BRIEF MENTION

Planning of Experiments. By D. R. Cox. Wiley, New York, 1958. vii+308 pp. $7.50.

While this book is not, in any sense, a mathematical or statistical book, it certainly provides worthwhile reading for the mathematician and statistician, as well as being an interesting reference for nonmathematically minded colleagues, who are always with us.

Theories of Figures of Celestial Bodies. By Wenceslas S. Jardetzky. Interscience, New York, 1958. xi+186 pp. $6.50.

A mathematical treatment of the shape of a rotating body.

Theoretical Mechanics: An Introduction to Mathematical Physics. By J. S. Ames and F. D. Murnaghan. Dover, New York, 1958. ix+462 pp. $2.00.

A republication of the 1929 volume by Ames and Murnaghan.

An Introduction to Fourier Methods and the LaPlace Transformation. By Philip Franklin. Dover, New York, 1958. x+289 pp. $1.75.

A corrected edition of the 1949 book *Fourier Methods* by the same author.

Introduction to Bessel Functions. By Frank Bowman. Dover, New York, 1958. x+135 pp. $1.35.

A reprinting of Bowman's book published in the late 1930's.

The Theory of the Potential. By William Duncan MacMillan. Dover, New York, 1958. xi+469 pp. $2.25.

A welcome republication of MacMillan's 1930 volume.

The Conduction of Heat in Solids. By H. S. Carslaw and J. C. Jaeger. Oxford University Press, New York 16, N. Y., 1959. viii+510 pp. $13.45.

This revised edition of the well-known standard by Carslaw and Jaeger will be welcomed by all. Call it to the attention of your engineering and physics faculty.

Dynamics and Nonlinear Mechanics. By E. Leimanis and N. Minorsky. Volume II, Surveys in Applied Mathematics. Wiley, New York, 1958. xii+206 pp. $7.75.

This book is divided into two portions, the first of which deals with recent advances in the dynamics of rigid bodies, celestial mechanics, and mathematical exterior ballistics, by Leimanis. The second portion by Minorsky deals with the theory of oscillations, including relaxation theory. Both portions conclude with up-to-date bibliographies.

Dictionary of Astronomy and Astronautics. By A. Spitz and F. Gaynor. Philosophical Library, New York, 1959. vi+439 pp. $6.00.

The understandable definitions in this volume are not limited to a few lines, but often run to half or even a full page discussion.

Vector Analysis with an Introduction to Tensor Analysis. By A. P. Wills. Dover, New York, 1958. xxxii+285 pp. $1.75.

A republication of Wills' 1931 text.

The Fundamental Principles of Quantum Mechanics with Elementary Applications. By Edwin C. Kemble. Dover, New York, 1958. xviii+611 pp. $2.95.

A reprinting of Kemble's 1937 expansion of an earlier work.

Quantum Electrodynamics. Julian Schwinger, Ed. Dover, New York, 1958. xvii+424 pp. pp. $2.45.

A collection of papers which appeared in various journals during the first half of the present century. Well selected, but unfortunately some of the print is too small for comfortable reading.

Statics and the Dynamics of a Particle. By William Duncan MacMillan. Dover, New York, 1958. xviii+430 pp. $2.00.

A reprint of MacMillan's 1927 book, which was originally designed as a text book in mechanics.

A Compendium of Mathematics and Physics. By Dorothy S. Meyler and O. G. Sutton. Van Nostrand, Princeton, N. J., 1958. x+384 pp. $5.00.

A collection of formulae from elementary mathematics, physics, and statistics.

Linear Groups with an Exposition of the Galois Field Theory. By Leonard Eugene Dickson. Dover, New York, 1959. xvi+312 pp. $1.95.

Since Dickson's book was written in 1900, Finite or Galois Field Theory has undergone extensive development. Many of the proofs given here could be materially shortened and the modern approach would be somewhat different. In spite of the work by Wedderburn, Chevalley, Dieudonné, and Artin, this work of Dickson's is still eminently readable and very much worthwhile. It could well be required reading for every beginning graduate student.

The Fourier Integral and Certain of Its Applications. By Norbert Wiener. Dover, New York, 1959. xi+201 pp. $1.50.

This reprint of the Cambridge University Press 1933 publication will be of particular interest to those who wish to study *Cybernetics* or *Time Series* by the same author, since much of the related mathematics is given here.

College Mathematics for Freshmen (2nd ed.). By P. K. Smith and H. F. Schroeder. Van Nostrand, Princeton, N. J., 1959. x+314 pp. $4.25.

This revision of the authors' 1948 work retains much of the flavor of the original book. The title is somewhat misleading, since the content is clearly remedial mathematics.

Lectures on the Theory of Elliptic Functions. By Harris Hancock. Dover, New York, 1958. xxiii+498 pp. $2.55.

A reprint of Hancock's 1909 book by the same title.

Elliptic Integrals. By Harris Hancock. Dover, New York, 1958. 104 pp. $1.25.

A reprint of Hancock's 1916 monograph.

The Foundations of Euclidean Geometry. By Henry George Forder. Dover, New York, 1958. xii+349 pp. $2.00.

The current interest in high school geometry has probably spurred the reprinting of this 1927 work.

The Elements of Non-Euclidean Geometry. By D. M. Y. Sommerville. Dover, New York, 1958. xvi+274 pp. $1.50.

A reprint of Sommerville's 1914 lectures.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

IBM COMBINATORIAL PROBLEMS INSTITUTE

An institute on Combinatorial Problems was sponsored by IBM Corporation at its Lamb Estate Research Center, near Ossining, New York, from July 6 to August 14,

1959. The institute was devoted to studies and seminars in mathematical programming, combinatorial design problems, game theory, and switching theory. The following accepted IBM's invitation to attend: Professor A. W. Tucker, Chairman, Princeton University, Dr. G. B. Dantzig, The RAND Corporation, Professor M. M. Flood, University of Michigan, Dr. D. R. Fulkerson, The RAND Corporation, Professor R. E. Gomory, Princeton University, Dr. A. J. Hoffman, General Electric Company, Dr. E. F. Moore, Bell Telephone Laboratories, Mr. J. Riordan, Bell Telephone Laboratories, Professor H. J. Ryser, Ohio State University, Professor C. B. Tompkins, UCLA.

The results of the Institute will be presented at a symposium, to be held at Princeton University in the fall of 1959, jointly sponsored by the Society for Industrial and Applied Mathematics and IBM Corporation.

NSF CONTINUES PROGRAMS

The National Science Foundation announces that it will continue to support the following programs during the summer of 1960 and the academic year 1960–61:

Undergraduate Research Participation
Research Participation for Teacher Training
Science Training for Secondary School Students

Colleges, universities, and other educational institutions desiring to consider participation in these programs may receive full information by addressing: Special Projects in Science Education, Scientific Personnel and Education, NSF, Washington 25, D. C.

PRELIMINARY ACTUARIAL EXAMINATION PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1959 Preliminary Actuarial Examination are as follows:

First prize of $200: Harold M. Stark, California Institute of Technology.

Additional prizes of $100 each: Stephen L. Alder, Harvard University; William G. Brown, University of Toronto; Richard M. Dudley, Harvard University; David D. Grossman, Harvard University; Bertrand I. Halperin, Harvard University; Ralph E. Miller, Harvard University; Steven A. Orszag, Forest Hills (N. Y.) High School; Barry Wolk, University of Manitoba.

The Society of Actuaries has authorized a similar set of nine prizes for the 1960 examination on Part 2.

The 1960 Preliminary Actuarial Examinations will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 11, 1960 and on November 16, 1960. The closing date for the May examinations is April 1, 1960, and for the November examinations, it is October 1, 1960. Further information concerning these examinations may be obtained from The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

AFFILIATE MEMBERSHIP IN IRE COMPUTER GROUP

Members of the Mathematical Association of America are cordially invited to affiliate with the Professional Group on Electronic Computers of The Institute of Radio Engineers. The Professional Group on Electronic Computers is an association of IRE members and affiliates with professional interest in the fields of analog and digital electronic computers. The group is concerned with the advancement of the electronic computer field and serves to aid in promoting close cooperation and exchange of technical information among its members.

The fee is $6.50 per year and includes the quarterly Transactions of this professional group. Application forms may be obtained from The Institute of Radio Engineers, 1

East 79th Street, New York 21, New York. Write attention of Mr. L. G. Cumming, Technical Secretary.

NEW JOURNAL

The Department of Commerce, through its Office of Technical Services (OTS), has started a new journal, TECHNICAL TRANSLATIONS. This journal appears twice a month and contains abstracts of foreign scientific publications (including mathematics) for which translations (in many cases from the Russian) are available through OTS. Further details may be obtained from Office of Technical Services, U. S. Department of Commerce, Washington 25, D. C.

PERSONAL ITEMS

Fresno State College: Professor Roy Dubisch will be on leave during the academic year 1959–60, and will be a Lecturer at the University of California, Berkeley; Professor G. D. Alkire will be Department Chairman (acting) during the absence of Professor Dubisch; Dr. W. L. Allen, Litton Industries, Santa Monica, California, Professor D. L. Boyer, New Mexico College of Agriculture and Mechanic Arts, and Dr. W. D. James, Stanford University, have been appointed Assistant Professors.

Seattle University: Dr. B. R. Toskey has been promoted to Assistant Professor; Dr. S. A. Husain, Purdue University, has been appointed Assistant Professor.

Associate Professor B. H. Arnold, Oregon State College, has been promoted to Professor.

Mr. P. L. Chessin, Westinghouse Electric Corporation, Baltimore, Maryland, has accepted a position as Applied Science Representative with the International Business Machines Corporation, Washington, D. C.

Mr. D. M. Estes, Baylor University, has accepted a position as Staff Assistant with the Southwestern Bell Telephone Company, Dallas, Texas.

Dr. T. S. Ferguson, University of California, Los Angeles, is on sabbatical leave and will spend the academic year at Princeton University.

Mr. F. W. Gibson, Douglas Aircraft Corporation, Santa Monica, California, has been appointed Graduate Research Mathematician with the Management Sciences Research Project, University of California, Los Angeles.

Dr. Seymour Ginsburg, National Cash Register Company, Hawthorne, California, has accepted a position as a member of the technical staff in the Information Processing Research Department of the Research Laboratories of the Hughes Aircraft Company.

Mr. R. B. Hall, Arizona State College, has accepted a position as Research Engineer with North American Aviation, Downey, California.

Associate Professor W. B. Houston, Jr., Morehouse College, has been appointed Assistant Professor at Carleton College.

Assistant Professor T. R. Jenkins, Washington State College, has accepted a position as Research Scientist with the Lockheed Aircraft Corporation, Missiles and Space Division, Sunnyvale, California.

Mr. J. C. Jones, University of Kentucky, has been appointed Teacher of Mathematics at the Cincinnati Public Schools.

Mr. P. J. Knopp, Harvard University, has been appointed Instructor at Spring Hill College.

Professor Stephen Kulik, Utah State University, has been appointed Professor at Long Beach State College.

Dr. Milton Lees, Institute for Advanced Study, has been appointed Visiting Assistant Mathematician at Brookhaven National Laboratories, Upton, Long Island, New York.

Assistant Professor Kenneth Loewen, Tabor College, has accepted a Graduate School Fellowship at The Pennsylvania State University for next year.

Mr. L. I. Lowell, Hughes Aircraft Company, Tucson, Arizona, has accepted a position as Research Engineer at the Missile Division of North American Aviation, Inc., Downey, California.

Dr. C. A. McCarthy, Yale University, has been appointed C. L. E. Moore Instructor in Mathematics at Massachusetts Institute of Technology.

Dr. Amin Muwafi, University of Florida, has been appointed Supervisor at the Ministry of Education, Amman, Jordan.

Mr. W. V. Neisius, Systematics, Inc., New York, has been promoted from General Manager to Vice-President.

Mr. C. C. Nielsen, Utah State Agricultural College, has accepted a position as Mathematician with the Hercules Powder Company, Magna, Utah.

Mr. Hugh Noland, Montana State College, has been appointed Assistant Professor at Chico State College.

Mr. G. W. Reitwiesner, Aberdeen Proving Ground, Maryland, has joined the Applied Mathematics Division of the National Bureau of Standards, Washington, D. C.

Dr. J. E. Wilkins, Jr., Nuclear Development Corporation of America, White Plains New York, has been promoted to Manager, Research and Development Operations.

Associate Professor Fred Brafman, University of Oklahoma, died February 4, 1959. He was a member of the Association for seven years.

Associate Professor Myrtle C. Brown, North Texas State College, died March 8, 1959. She was a member of the Association for thirty-six years.

Professor Emeritus E. R. Smith, Florida State University, died March 21, 1959. He was a member of the Association for thirty-six years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

Professor H. M. Gehman, Secretary Treasurer, announces that the following 121 persons have been elected to membership by the Board of Governors on applications duly certified.

EMIL AMELOTTI, M.S. (Illinois) Chairman of Dept., Villanova University

CAPT. EDWARD ANLIAN, M.S. (S.U. of Iowa) Instr., United States Air Force Academy

PAUL ANTHONY, M.A.B. (New York) Professor, New York City Community College

PASQUALE J. ARPAIA, B.S. (St. John's) Teacher, Woodland Junior High School, East Meadow, New York

RICHARD A. BAKER, Student, Portland State College

JOE R. BALLARD, B.A. (North Texas S.C.) Mathematician, National Bureau of Standards

WILLIAM C. BECK, M.S. (S.U. of Iowa) Instr., Wisconsin State College

VICTOR BIK, B.S. (DePaul) Teacher, St. Stanislaus High School, Cleveland, Ohio

MRS. LOIS BLONDEAU, Student, Wisconsin State College

ALFRED BLUMSTEIN, M.A. (Buffalo) Research Physicist, Cornell Aeronautical Lab.

EDWARD E. BOSMAN, M.A. (Columbia) Head of Dept., Barrett School, Cresco, Pennsylvania

HELEN M. BOWDEN, M.A. (Wisconsin) Teacher, Bay Ridge High School, Brooklyn, New York

ALFRED G. BOYD, B.S. in E.E. (North Dakota) Manager, Marketing Planning, General Electric

M. PHILBRICK BRIDGESS, A.M. (Harvard) Head of Dept., Roxbury Latin School, Massachusetts

BROTHER ALEXIS H. GUILBAULT, M.S. (Montreal) Instr., Prevost High School, Fall River, Massachusetts; Grad. Student, Columbia University

BROTHER JOSEPH HEISLER, B.S. (St. Edward's) Grad. Student, University of Notre Dame

JOHN BROWN, Student, McGill University

KENNETH N. BROWN, Student, Los Angeles State College

ARTHUR F. BROWNELL, JR., B.S. in Ed. (Northern Illinois) Grad. Asst., Mankato State College

ROBERT L. CAMPBELL, M.Ed. (Louisiana S.U.) Instr., University of Hawaii

WALTER J. CARPENTER, M.A. (North Carolina) Asso. Professor, North Georgia College

JAMES W. CEDERBERG, Student, University of Kansas

JOHN C. CHITWOOD, JR., M.A. (Texas) Asst. Professor, University of Kansas City

AMBROSE R. CLARKE, Ed.M. (St. Lawrence) Head of Dept., Batavia High School, New York

WALLACE A. COLE, M.S. (Wisconsin) Asst. Professor, Michigan College of Mining and Technology

MARTHA J. COLQUITT, A.B. (Mercer) Teacher, A. L. Miller Junior High School, Macon, Georgia

ROBERT E. COMLEY, M.S. (N.Y.S.T.C., Albany) Instr., Hudson Valley Technical Institute, Troy, New York
ROBERT M. COTTON, Student, Swarthmore College
JOSE B. COURTIS, Actuary (Facultad de Ciencias Economicas) IBM World Trade Corp., Buenos Aires, Argentina
NOEL E. CUFF, M.S. (Kentucky) Asso. Engineer, The Martin Co.
CAROL CUMMINGS, Student, Texas Technological College
ARLEY T. CURLEE, M.A. (North Carolina) Professor and Head of Dept., Salem College
EVAN L. DAVIS, JR., Student, University of Colorado
KENNETH J. DAVIS, Student, Wofford College
ERNEST J. ECKERT, M.A. (Southern California) Instr., Los Angeles State College
DON M. ESTES, Student, Baylor University
CHARLES F. FEDERSPIEL, M.A. (Michigan) 1509 St. Mary's Street, Raleigh, North Carolina
JOHN A. FIGUEROA, Student, Los Angeles State College
DONALD E. FITZGERALD, A.M. (Yale) Instr., Long Beach City College
EILEEN G. FLAHERTY, A.B. (Salve Regina) Instr., Salve Regina College
MICHAEL A. GERAGHTY, Ph.D. (Notre Dame) Jr. Research Instr., University of Notre Dame
MRS. ALVA M. GOLDSMITH, B.A. (Smith) Teacher, Charles E. Gorton High School, Yonkers, New York
MRS. RICHARD H. GRAMANN, B.S. (Illinois) Teacher, Arlington Township High School, Illinois
HARVEY W. GREENE, M.A. (Missouri) Mathematician, Jersey Production Research Co.
JOHN R. HALEY, Student, University of Tulsa
J. HARRISON HANCOCK, M.S. (Brown) Mathematician, Naval Research Lab.
HARRY R. HANNUM, A.B.A. (Pennsylvania) Research Manager, The Philadelphia Inquirer, Pennsylvania
HARVEY A. HARTENSTEIN, Student, Long Beach State College
LEONARD J. HASSLER, B.S. (Yale) Master, Pomfret School, Connecticut
JANE M. HILL, M.A. (Ohio S.U.) Teacher, Washington, D.C. Public Schools
NICK HOLONYAK, JR., Ph.D. (Illinois) Physicist, General Electric, Syracuse, New York
JOHN H. HUNTZINGER, M.S. (Temple) Head of Dept., Northeast High School, Philadelphia, Pennsylvania
JURI KALVISTE, B.S. in E.E. (Washington) Research Engineer, Boeing Airplane Co.
ARTHUR KLEIN, Ed.M. (Boston T.C.) Master, Boston Latin School
LYNN C. KURTZ, Student, South Dakota School of Mines and Technology
JOHN P. LASCHENSKI, Student, College of the Holy Cross
MARTHA LENTZ, Student, Wisconsin State College
HOWARD LEVI, Ph.D. (Columbia) Asso. Professor, Columbia University
F. A. LIMOUZIN, Professor Emeritus, Ecole Normales, France
MARY E. LOGAN, Student, University of Kentucky
ROBERT C. LYNESS, B.A. (Oxford) Inspector of Schools, London, England
LAVAL MATHIEU, Student, University of Montreal
MOTHER M. CARMELITA MCINTYRE, Ed.M. (St. Louis) Instr., Springfield Junior College
LEONARD B. MCKEE, B.S. (California) Senior Research Chemist, Shell Chemical Corp.
KARL MICHAELIS, Dipl.-Phys. (Hansische) Teacher, Heeresoffizierschule II, Hamburg, Germany
JOHN W. MILSOM, B.A. (Pennsylvania S.U.) Grad. Student, Southwestern Louisiana Institute
DAVID B. MOSER, Student, Emory University
PEGGY J. MULLINS, Student, University of Alabama
FRANCIS X. MULVIHILL, A.B. (Boston Coll.) Lt. (jg), Naval Amphibious Base, Virginia
SANDRA G. NESS, Student, Emory University
NATHAN O. NILES, M.S. (St. Lawrence) Asso. Professor, Naval Academy
HELEN PAISNER, B.A. (Hunter) Lecturer, Barnard College; Grad. Student, Columbia University
STEPHEN D. PASTOR, Student, Oberlin College
RUSI K. N. PATELL, M.A. (Columbia) Research Asst., Columbia University
GEORGE B. PECK, M.S. (Illinois) Asst. Professor, Arizona State University
WILLIAM PENCHUCK, B.Ed. (Alberta) Vice Principal, Ft. Assinboine School, Alberta, Canada
RICHARD W. PIKE, Student, Wisconsin State College
HOWARD L. PROUSE, M.S. (Wayne S.T.C.) Supervisor, Math. Dept., Mankato State College
NORMAN J. PULLMAN, M.A. (Harvard) Grad. Asst., Syracuse University
CHARLES W. RAY, B.A. (San Francisco S.C.) Grad. Student, San Francisco State College
WILLIAM H. RICHARDSON, Student, Chico State College
H. ARNALL ROBERTS, B.E. (Vanderbilt) Grad. Student, Emory University
PAUL ROETHEL, Student, Wisconsin State College
JAMES H. ROLF, Student, Eastern Kentucky State College
EDGAR A. ROSE, Ed.M. (Rochester) Teacher, Monroe High School, Rochester, New York
MRS. DOROTHY S. RUTLEDGE, B.A. (Birmingham-Southern) Grad. Student, Emory University
JOSEPH E. RYUS, B.A. (California, Berkeley) Librarian, University of California
JOHN C. SALLIS, Student, University of Arkansas
JOHN F. H. SCHLUEP, M.A. (Columbia) Head of Dept., Cato Meridian Central School, New York
DAVID I. SCHNEIDER, Student, Oberlin College
HILBERT SCHULTZ, Student, Wisconsin State College
NATHAN T. SEELY, JR., M.A. (Pennsylvania) Acting Chairman of Dept., Agricultural and Technical College of North Carolina
CARL J. SINKE, Ph.D. (Purdue) Asst. Professor, Calvin College
SISTER CATHERINE JOSEPHINE, Ph.D. (Boston) Asso. Professor, Emmanuel College
PARKY SKELTON, Student, Emory University
W. RICHARD SLINKMAN, M.S. (Kansas S.C., Pittsburg) Instr., University of Colorado
JAMES M. SLOSS, B.S. (Pomona) Research Asst., University of California, Berkeley
CHARLES J. SMITH, A.B. (Washington) Grad. Student, Washington University
GEORGE M. SNEED, Student, Emory University
BERNARD SOHMER, Ph.D. (New York) Instr., College of the City of New York
FRANK W. STARKS, III, Student, J. M. Atherton High School, Louisville, Kentucky
WILLIAM F. STEELE, Asst. Professor, Heidelberg College
ROBERT M. STONE, M.Ed. (Howard Payne) Asst. Professor, Howard Payne College
EDWARD E. SUNDELL, B.A. (S.C. of Washington) Math. Analyst, General Electric Co.
JAMES J. SWISHER, JR., M.B.S. (Colorado) Instr., College of the Sequoias
DONALD G. TEMPLETON, Student, University of Chattanooga
MRS. JUANITA S. TOLSON, M.S. (Howard) Chairman of Dept., Roosevelt High School, Washington, D.C.
WILLIAM F. TYNDALL, B.S. (Franklin & Marshall) Grad. Student, Brown University
HAROLD R. VANDENBURGH, B.A. (North Texas S.C.) Mathematician, Project Matterhorn, Princeton University
ROBERT G. VAN METER, A.M. (Duke) Asst. Professor, Geneva College
JOHN R. VEATCH, Student, Whitman College
WILLIAM M. WAGNER, M.S. (Iowa S.C.) Engineer-Computer, North American Aviation
NORMAN H. WALTON, Student, San Jose State College
ULYSSES V. WARD, M.A. (Wayne State) Instr., Howard University
RONSON J. WARNE, M.S. (New York) Asst., University of Tennessee
JOHN WEISSMAN, M.A. (Temple) Instr., Rutgers University
MARVIN J. WINER, Student, University of Buffalo
BURTON M. WOODWARD, Student, University of Florida
WILLIAM S. WUNCH, Ph.D. (Stanford) Asso. Professor, Arizona State University
FREDERIC A. WYATT, B.A. (Union) Senior Consultant, Richardson, Bellows, Henry & Co.
LILLIAN ZARLING, M.A. (Minnesota) Asso. Professor, Wisconsin State College

THE APRIL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at Centre College, Danville, Kentucky, on April 25, 1959. Professor W. J. Robinson, Chairman of the Section, presided at both the morning and afternoon session. Fifty-three persons attended the meeting including forty-three members of the Association.

At the business meeting the following officers were elected for the coming year: Chairman, Professor W. C. Royster, University of Kentucky; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor T. J. Pignani, University of Kentucky.

The following papers were presented:

1. *The numerical solution of the Van der Pol equation*, by Professor W. Krogdahl, University of Kentucky, introduced by the Secretary.

Limit cycle solutions of the Van der Pol equation were obtained by numerical integration for integral values of the parameter from 1 to 10. The characteristics of these solutions were compared with the characteristics of solutions for large parameter values as determined by asymptotic formulae.

2. *Singularities of analytic functions*, by Mr. J. P. King, University of Kentucky.

The author uses the transformation $z=t+kt^2$ to obtain new criteria that the point $z=1$ be a singular point for $f(z)=\sum a_n z^n$.

3. *Some remarks on grading*, by Professor A. G. Anderson, Western Kentucky State College.

A scheme whereby much of the arbitrariness in grading can be eliminated is the subject of the paper. The method is not designed for experienced teachers or advanced courses but should enable a new department member to conform to an institution's grading policy while at the same time incorporating ideas of his own.

4. *A class of odd typically real functions in an ellipse*, by Professor W. C. Royster, University of Kentucky.

Let T_0 denote the class of odd functions $(*) f(z)=z+\sum_{n=1}^{\infty} a_{2n+1}T_{2n+1}(z)$, $T_n(z)=\cos(n\cos^{-1}z)$, with the properties that (a) $f(z)$ is regular in the ellipse with foci ± 1 and semi-axes $a>b>0$, (b) $f(z)$ is real on the real axis and (c) if z lies in E then $\operatorname{Im}\{z\}\cdot\operatorname{Im}\{f(z)\}\geq 0$ and $\operatorname{Re}\{z\}\cdot\operatorname{Re}\{f(z)\}\geq 0$. It is shown that if $f(z)$ given by $(*)$ belongs to the class T_0 then $|a_{2n+1}R^{2n}+a_{2n-1}R^{-2n}|\leq 2$, $n=1, 2, \cdots$.

5. *A problem in musical composition*, by Professor W. H. Spragens, University of Louisville.

In this paper the author studies problems of intervals as related to the twelve tone scale.

6. *Discrete analytic functions*, by Mr. W. T. Sledd, University of Kentucky.

An expository talk.

7. *Report on the Visiting Lecturer Program*, by Professor S. E. Pence, University of Kentucky.

8. *Remarks on MacIntyre's method of solving a Pfaffian*, by Professor T. J. Pignani, University of Kentucky.

An expository talk concerned with MacIntyre's (Proc. Edinburgh Math. Soc. vol. 4, 1934–1936) method of solving the exact differential equation $P(x, y)dx+Q(x, y)dy=0$.

9. *Some problems in constructive function theory*, by Professor George Piranian, University of Michigan. (Invited one-hour address).

V. F. COWLING, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section of the Mathematical Association of America was held April 17–18, 1959 at the University of Texas, Austin, Texas. There were 150 in attendance, including 100 members of the Association.

The following officers were elected for the coming year: Chairman, Professor W. T. Guy, Jr., University of Texas; Vice-Chairman, Professor P. R. Culwell, San Antonio College; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University. The next meeting will be held at San Antonio College, San Antonio, Texas.

Professor H. J. Ettlinger gave a detailed report of the High School Mathematics Contest which was given on a trial basis. Through Professor Ettlinger's expert leadership, the results were very satisfactory and the section decided to continue the contest next year.

The following program was presented:

1. *Diagonal forms of odd prime degree over finite fields*, by Professor J. F. Gray, St. Mary's University of San Antonio, introduced by the Secretary.

Consider the problem of representing zero by a non-trivial linear combination of pth powers in a finite field k. By extending a theorem of D. J. Lewis, it is proved that p pth powers suffice when p is an odd prime. If, in addition, $p \geq 5$, then $p-1$ pth powers are shown to suffice and, for $p \geq 7$, $p-2$ pth powers are sufficient. In general, where p and p_0 are odd primes and $p \geq p_0$, there exists an integer $t(p_0)$ such that over a finite field k, $a_1x_1^p + a_2x_2^p + \cdots + a_{p-t(p_0)}x_{p-t(p_0)}^p = 0 (a_i \in k)$ has a nontrivial solution in k. The values $t(3)=0$ and $t(5)=1$ are best possible. That $p_0 - t(p_0) > \log_2 p_0 + 1$ for an infinity of primes p_0 would follow from an undoubted, but unverified, conjecture of Dickson's which implies the existence of an infinity of prime pairs (p, q) with $q = 2p+1$.

2. *Some mathematical aspects of the problem of stability*, by Professor W. S. McCulley, Agricultural and Mechanical College of Texas.

Consideration of an electro-mechanical system described by a second order linear ordinary differential equation with constant coefficients indicated that negative real parts for the roots of auxiliary equation represent stability. For feedback systems, information about stability may be obtained by Routh-Hurwitz criterion, Bode plots, Nyquist criterion, root locus method, describing functions, or closed loop pole-zero location.

3. *The accumulation of round-off error in the numerical integration of ordinary differential equations*, by Professor Peter Henrici, University of California, Los Angeles. (By invitation.)

The paper studies the dependence of the accumulated round-off error (r.o.e.) in the numerical integration of an ordinary differential equation by a general multi-step method in the sense of Dahlquist (Math. Scand., vol. 4, 1956, pp. 33–53) on the local r.o.e. Under the assumption that the local r.o.e. are independent random variables with known distributions, a formula for the variance of the accumulated r.o.e. is derived. The results are tested by numerical experiments in which a given differential equation is solved by the same method under varying initial conditions.

4. *An overall look at extended analytic geometry*, by Professor R. S. Underwood, Texas Technological College.

When X and Y are particular single-valued functions of n variables, a locus on the XY-plane (occasionally vacuous) is implicitly associated with any equation in the variables. Extended analytic geometry provides ways of finding these loci rapidly by use of algebra, plane analytic geometry, and in some cases, calculus. The system yields solutions, or proves inconsistence, for various Diophantine and simultaneous equations, including all linear-versus-quadratic pairs. Trick results appear, as well as properties of equations that are independent of plotting rules. Criteria for quadric surfaces shorter than the conventional ones are developed.

5. *Extended quadrics of Darboux*, by Professor Dale Maness, Howard Payne College.

The equation of the most general non-composite quadric hypersurface having contact of second order at a general point with an analytic hypersurface in n-space is found. The classical tangents of Daroux are found to be specializations of cubic $(n-2)$-dimensional cones to be known as the "cones of Darboux." The classical directions of Darboux are generalized for hypersurfaces and are to be known as the "extended directions of Darboux." A one parameter family of hyperquadrics to be known as the "extended quadrics of Darboux" is defined and specializes to give the Darboux family of quadric surfaces of three space.

6. *The unitary equivalence of 3×3 matrices*, by Professor Carl Pearcy, Rice Institute, introduced by the Secretary.

Consider the collection $A^{n_1}A^{n_2}\cdots A^{n_j}$, where $n_1, n_2, \cdots n_j$ is any finite sequence of non-negative integers, and A is any $n\times n$ complex matrix. Specht showed that the traces of the above collection form a complete set of unitary invariants for $n\times n$ complex matrices. Murnaghan showed that for $n=2$ a subset of three of the above traces suffices, and the author has shown that for $n=3$ a subset of eleven traces suffices.

7. *Numerical integration with weight functions X and X^2*, by Professor R. E. Greenwood, University of Texas.

The use of the correspondence $\int_a^b x^k f(x)dx \sim \sum_{i=1}^{n} x_{i,n}^k f(x_{i,n})$ is advantageous in evaluating first moments of means $(k=1)$ and second moments $(k=2)$ for certain empirically determined functions. The set of needed abscissas $\{x_{i,n}\}$, $i=1, \cdots, n$, may be called a set of generalized theorem-of-the-mean points. It is shown that for some values of n one or more of these points may fall outside of the interval (a, b) thereby precluding effective use of the numerical approximation correspondence. Other values of n yield usable results. This work is a portion of an article by Greenwood, Carnahan and Nolley which is scheduled to appear in a forthcoming issue of *Mathematical Tables and Other Aids to Computation*.

8. *Inversion of Jacobian matrices*, by Professor Louis Brand, University of Houston.

In a transformation of coordinates $x^i=f^i(y^1, \cdots, y^n)$, $i=1, \cdots, n$, the n^2 derivatives $\partial y^i/\partial x^j$ may be computed from the fact that the Jacobian matrices $(\partial x^i/\partial y^j)$ and $(\partial y^i/\partial x^j)$ are reciprocals when written so that i is a row index. The inversion of $(\partial x^i/\partial y^j)$ is particularly simple when the transformation is from one orthogonal system to another. Then $(\partial x^i/\partial y^j)$ can be expressed as the product MD of an orthogonal matrix M and a diagonal matrix D, whose elements are the lengths of its column vectors; and $(\partial y^i/\partial x^j)=(MD)^{-1}=D^{-1}M'$ where M' is the transpose of M. The calculation may also be made by observing that the vectors $e_{mi}=(\partial x^1/\partial y^i, \cdots, \partial x^n/\partial y^i)$, $e_m^j=(\partial y^j/\partial x^1, \cdots, \partial y^j/\partial x^n)$ satisfy $e_i\cdot e^j=\delta_i^j$ and hence form reciprocal sets. Thus $e_m^j=g^{jr}e_{mr}$; or since $g^{jk}=0(j\neq K)$, $g^{jj}=1/g_{jj}$ for orthogonal transformations, $e_m^j=g^{jj}e_{mj}=e_{mj}/g_{jj}=e_{mj}/(e_j\cdot e_j)$ with summation suspended.

9. *Sub-structure vs. super-structure in presenting real algebra on the secondary and elementary collegiate level*, by Professor J. F. Gray, St. Mary's University of San Antonio.

In the current concern to present the real number system, both on the secondary and on the elementary collegiate level, from a truly mathematical point of view, two approaches stand out. The first proceeds from structure to substructure, beginning with a set of postulates, assumptions, or properties of the real number system essentially equivalent to the axioms for an ordered field and later descending to study the various sub-systems of interest. The alternative approach proceeds from structure to super-structure, ascending from a consideration of the natural numbers to the rational and real fields. Reasons and supporting experiences are cited to strongly suggest that the first approach is far more desirable.

10. *Equilibrium position of a certain flexible cable*, by Professor Guy Johnson, Rice Institute.

11. *Concerning matrix norms*, by Professor Ben Fitzpatrick, Jr., University of Texas.

12. *Peculiar sort of power series*, by Professor G. R. MacLane, Rice Institute.

13. *Coverings*, by Mr. Terence Reed, Rice Institute, introduced by the Secretary.

14. *Advanced calculus for undergraduate majors*, by Dr. Fred Bamforth, Texas Christian University, introduced by the Secretary.

C. R. SHERER, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The twelfth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the University of Oregon, Eugene, Oregon on June 19, 1959 in conjunction with the 558th meeting of the American Mathematical Society and jointly with the Pacific Northwest Section of the Society for Industrial and Applied Mathematics. Professor Kenneth Bush, chairman of the Section, presided over the meetings. One hundred and eleven persons were in attendance, including 69 members of the Association.

At the Business Meeting the following officers were elected: Chairman, Professor A. E. Livingston, University of Washington; Vice-Chairman, Professor T. G. Ostrom, University of Montana; Secretary-Treasurer, Professor K. S. Ghent, University of Oregon.

Professor Allendoerfer reported briefly on plans of the Mathematical Association of America.

The program was as follows:

1. *Set theory and numbers*, by Professor J. L. Kelley, University of California, Berkeley (by invitation).

The purpose of the paper is to construct the real number system within intuitive set theory. The void set is 0, and for each set x we let x' be the set whose only members are x and the members of x. (*i.e.*, $y \in x'$ iff $y \in x$ or $y = x$). An inductive class Δ is a class such that $0 \in \Delta$ and if $x \in \Delta$ then $x' \in \Delta$. A set n is a natural number iff n belongs to each inductive class. The Peano postulates are easily proved. Using the properties of the natural numbers as constructed, addition and multiplication are defined as cardinal addition and multiplication. The development is simple and intuitively natural.

2. *Boundary value problems in ordinary differential equations*, by Professor René De Vogelaere, University of California, Berkeley, introduced by the Secretary.

A general method of numerical analysis, called the *discretization method* is defined using as illustration the problem of integrating a function over an interval. The definition, inspired by work of Kantorovich, Kelley, Lemaître and Crout, contains as special cases a large number of the methods used for the solution of differential equations, ordinary and partial, as well as integral equations; in particular, difference, Galerkin and least-square methods. The compactness of presentation that this definition permits is illustrated for the solution of boundary value problems of differential equations. Advantage of restricting the functions involved to non-equidistant points over the prevailing practice (difference methods) is stressed.

3. *An elementary approach to Lebesgue integration*, by Professor H. H. Schaefer, State College of Washington.

This is a brief elementary exposition of a basically well-known method to obtain the concept of Lebesgue integral for, say, real functions on a compact interval $I \subset R^n$. Here "elementary" means that no use is made of the notions of measure and measurable function; familiarity with the Riemann integral and with the completion of a metric space is assumed. The (didactical) stress is laid upon the fact that with this material, along with the basic tools of calculus and using a concept equivalent to (Lebesgue) measure 0, the most important properties of the L-integral can be deduced including the convergence theorems of B. Levi, Fatou, and Lebesgue. The method extends easily to the Lebesgue-Stieltjes integral and shows rather clearly the relation to the corresponding Riemann integrals.

4. *Some mathematical aspects of the theory of life contingencies*, by Professor S. A. Jennings, University of British Columbia.

The theory of life contingencies offers a variety of interesting and important applications of elementary calculus which many beginning students might find more meaningful than the conventional discussions of velocity, acceleration, centroids and moments of inertia. To illustrate this contention, the speaker discussed how, starting with ${}_tp_x$, the probability that a person aged x will survive to age $x+t$, the force of mortality may be defined as a logarithmic derivative, and Gompertz' and Makeham's laws of mortality derived by solving a differential equation. Of more interest from the point of view of calculus applications, however, are problems involving integration. The speaker mentioned several examples of this type.

5. *Mathematics goes to market*, by Professor R. E. Gaskell, Oregon State College.

Interest of the government in mathematics and mathematicians developed from military necessity between 1940–1945, and this has carried over to industry since then. There are, as a result, growing numbers of mathematicians employed in industry, in fact the growth could be described as explosive. The dramatic effect of high-speed computing machinery is easily visible, but fewer people—mathematicians as well as non-mathematicians—realize the no less dramatic effect of new developments in mathematics itself on industry. A need for more mathematicians is becoming evident, but beyond this more mathematics will be required of all of us—producers, consumers, and even those few who insist that they are merely observers.

K. S. GHENT, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary:

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA

IOWA, State University of Iowa, Iowa City, October 16, 1959.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April, 1960.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, American University, Washington, D. C., December 5, 1959.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY, Princeton University, November 7, 1959.

NORTHEASTERN, Boston College, Chestnut Hill, Massachusetts, November 28, 1959.

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA, Oklahoma City University, October 23, 1959.

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA, University of Delaware, Newark, November 28, 1959.

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.

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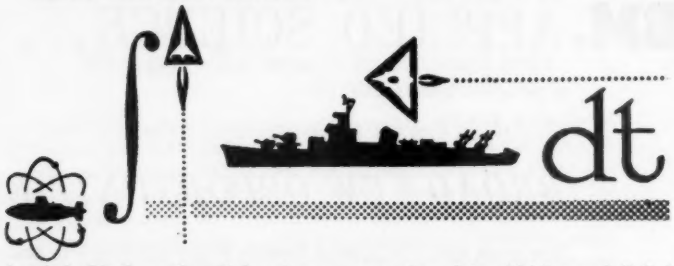

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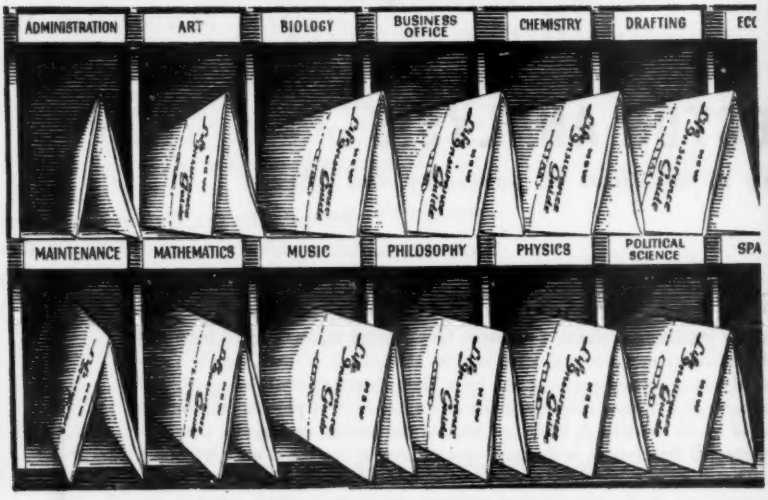

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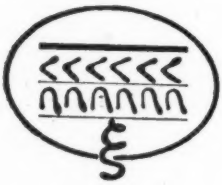

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